

\mathbb{C}^* -actions on generalized Calogero-Moser spaces and Hilbert schemes

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Abstract

In this paper we study \mathbb{C}^* -actions on generalized Calogero-Moser spaces and Hilbert schemes. The spectrum of the centre of the rational Cherednik algebra associated to the complex reflection group $(\mathbb{Z}/l\mathbb{Z}) \wr S_n$ at $t = 0$ is isomorphic, via the Etingof-Ginzburg map, to a certain Nakajima quiver variety. Assuming smoothness, reflection functors yield a hyper-Kähler isometry between this quiver variety and a subvariety of a Hilbert scheme. We show that the induced map on the labelling sets of \mathbb{C}^* -fixed points is given by a version of the classical bijection between l -multipartitions and partitions with a certain fixed core depending on the choice of parameters, as claimed by Gordon in [10]. We apply this result to obtain a new proof and a generalization of the q -hook formula.

1 Introduction

1.1 The background. It is well known that there is a connection between rational Cherednik algebras, Nakajima quiver varieties and Hilbert schemes. This connection manifests itself in several ways. Firstly, consider the rational Cherednik algebra $\mathbb{H}_{0,\mathbf{h}}(\Gamma_n)$ associated to the complex reflection group $\Gamma_n := (\mathbb{Z}/l\mathbb{Z}) \wr S_n$ at $t = 0$ and a generic parameter \mathbf{h} . The Etingof-Ginzburg map yields an isomorphism of schemes between the spectrum $\text{Spec } Z_{0,\mathbf{h}}$ of the centre of $\mathbb{H}_{0,\mathbf{h}}(\Gamma_n)$ and a certain Nakajima quiver variety $\mathcal{X}_{\theta_{\mathbf{h}}}(n\delta)$ generalizing the classical construction of the Calogero-Moser space by Wilson in [29]. Moreover, this quiver variety, considered as a hyper-Kähler manifold, admits a non-algebraic embedding into a Hilbert scheme. This embedding is constructed using Nakajima reflection functors.

Secondly, there is also a relationship between Hilbert schemes and the rational Cherednik algebra $\mathbb{H}_{1,\mathbf{h}}(S_n)$ at $t = 1$. Let $U_{1,\mathbf{h}}(S_n) := e\mathbb{H}_{1,\mathbf{h}}(S_n)e$ denote the spherical subalgebra of $\mathbb{H}_{1,\mathbf{h}}(S_n)$. Using the machinery of \mathbb{Z} -algebras, Gordon and Stafford constructed in [11] a functor from the category of modules over $U_{1,\mathbf{h}}(S_n)$ to the category of coherent sheaves on a certain version of the Hilbert scheme of n points in the plane. Their results can be interpreted as saying that $U_{1,\mathbf{h}}(S_n)$ constitutes a noncommutative deformation of the homogeneous coordinate ring of the Hilbert scheme. The generalization of these results from S_n to Γ_n was established by Gordon in [10]. Similar results were also obtained by Kashiwara and Rouquier in [14] using microlocalization and W -algebra methods.

Thirdly, there is a connection between the category $\mathcal{O}_{\mathbf{h}}$ of $\mathbb{H}_{1,\mathbf{h}}(\Gamma_n)$ and the geometry of a certain quiver variety $\mathcal{M}_{2\theta_{\mathbf{h}}}(n\delta)$ associated to the extended cyclic quiver. This quiver variety yields a symplectic resolution of the singular variety \mathbb{C}^{2n}/Γ_n . In [10] Gordon defined a geometric ordering $\prec_{\mathbf{h}}$ on the set $\mathcal{P}(l, n)$ of l -multipartitions of n using the closure relations between attracting sets of \mathbb{C}^* -fixed points in $\mathcal{M}_{2\theta_{\mathbf{h}}}(n\delta)$. This geometric ordering is refined by the c -ordering, which is known to define a highest weight structure on the category $\mathcal{O}_{\mathbf{h}}$. Gordon conjectured that the geometric ordering also gives a highest weight structure on $\mathcal{O}_{\mathbf{h}}$.

1.2 The problem. Let $\mathcal{P}_{\varnothing}(nl)$ denote the set of partitions of nl with a trivial l -core. It is well known that the map sending a partition in $\mathcal{P}_{\varnothing}(nl)$ to its l -quotient defines a bijection between $\mathcal{P}_{\varnothing}(nl)$ and $\mathcal{P}(l, n)$. We will refer to it as the l -quotient bijection. Given a multicharge \mathbf{h} or,

equivalently, an element of the affine symmetric group \tilde{S}_l , this bijection can be generalized to a bijection between $\mathcal{P}_\nu(K)$ and $\mathcal{P}(l, n)$ (see e.g. [10, §6]). Here ν is an l -core depending on \mathbf{h} , $K = nl + |\nu|$ and $\mathcal{P}_\nu(K)$ is the set of partitions of K with l -core ν . We will refer to such a generalized bijection as the \mathbf{h} -twisted l -quotient bijection. Let $\prec'_\mathbf{h}$ be the partial order on $\mathcal{P}(l, n)$ induced from the dominance order on $\mathcal{P}_\nu(K)$ under the \mathbf{h} -twisted l -quotient bijection. We call it the *combinatorial ordering*. One of the central claims of [10] is:

Claim A. *The geometric ordering $\prec_\mathbf{h}$ coincides with the combinatorial ordering $\prec'_\mathbf{h}$ on $\mathcal{P}(l, n)$.*

The main goal of the present paper is to prove Claim A. We remark that the proof given in [10] is incorrect because it relies on a false assumption about the c -order being total inside c -chambers (see §11.2 below for a counterexample). Claim A has several important applications. In [6] Dunkl and Griffeth proved that the combinatorial ordering $\prec'_\mathbf{h}$ defines a highest weight structure on $\mathcal{O}_\mathbf{h}$. Our result connects this highest weight structure, through the geometric ordering, with the geometry of quiver varieties. This was, in fact, the original motivation behind Gordon's paper [10]. The results of Dunkl and Griffeth together with Claim A imply a strengthening of Rouquier's theorem ([26, Theorem 5.5]) regarding equivalences between categories $\mathcal{O}_\mathbf{h}$ for different parameters \mathbf{h} . They also imply that the nonzero images of simple modules in $\mathcal{O}_\mathbf{h}$ under the Knizhnik-Zamolodchikov functor form a canonical basis set for the finite-dimensional Hecke algebra $\mathcal{H}_q(\Gamma_n)$. Finally, we remark that Claim A was used in the proof of Haiman's wreath Macdonald positivity conjecture by Finkelberg and Bezrukavnikov (see [1], especially Lemma 3.8).

1.3 A geometric interpretation. Claim A can be interpreted in terms of the geometry of generalized Calogero-Moser spaces and Hilbert schemes. As we have already mentioned, the Etingof-Ginzburg isomorphism composed with reflection functors yields a non-algebraic embedding of $\text{Spec } Z_{0,\mathbf{h}}$ into $\text{Hilb}(K)$, the Hilbert scheme of K points in \mathbb{C}^2 . Let us consider this embedding in more detail. Etingof and Ginzburg showed in [7] that $\text{Spec } Z_{0,\mathbf{h}}$ is isomorphic as a scheme to a certain Nakajima quiver variety $\mathcal{X}_{\theta_\mathbf{h}}(n\delta)$, whose construction we review in section 8. We will always assume that the parameter $\mathbf{h} \in \mathbb{Q}^l$ is chosen so that our quiver variety is smooth. The scheme $\mathcal{X}_{\theta_\mathbf{h}}(n\delta)$ can also be endowed with the structure of a hyper-Kähler manifold. Using the reflection functors defined by Nakajima in [23] one can construct a hyper-Kähler isometry $\mathcal{X}_{\theta_\mathbf{h}}(n\delta) \rightarrow \mathcal{X}_{-1/2}(\gamma)$, where γ is a dimension vector dependent on \mathbf{h} . The rotation of complex structure yields a diffeomorphism between $\mathcal{X}_{-1/2}(\gamma)$ and a certain GIT quotient $\mathcal{M}_{-1}(\gamma)$. Let $\text{Hilb}(K)^{\mathbb{Z}/l\mathbb{Z}}$ denote the $\mathbb{Z}/l\mathbb{Z}$ -invariants in $\text{Hilb}(K)$. The scheme $\mathcal{M}_{-1}(\gamma)$ is isomorphic to an irreducible component of $\text{Hilb}(K)^{\mathbb{Z}/l\mathbb{Z}}$. All the maps involved are summarized in the following diagram:

$$\text{Spec } Z_{0,\mathbf{h}} \xrightarrow{\text{EG}} \mathcal{X}_{\theta_\mathbf{h}}(n\delta) \xrightarrow{\text{RF}} \mathcal{X}_{-1/2}(\gamma) \xrightarrow{\text{Rot}} \mathcal{M}_{-1}(\gamma) \hookrightarrow \text{Hilb}(K)^{\mathbb{Z}/l\mathbb{Z}}. \quad (1)$$

Both $\text{Spec } Z_{0,\mathbf{h}}$ and $\text{Hilb}(K)$ are endowed with \mathbb{C}^* -actions. The closed fixed points under these actions are naturally labelled by l -multipartitions of n in the first case and partitions of K in the second case. Let us explain the labelling of the fixed points in more detail. Gordon showed in [9] that the closed \mathbb{C}^* -fixed points in $\text{Spec } Z_{0,\mathbf{h}}$ are precisely the annihilators of the simple modules over the restricted rational Cherednik algebra. The latter arise as quotients of baby Verma modules. Since baby Verma modules are induced from simple representations of the generalized symmetric group Γ_n , the closed \mathbb{C}^* -fixed points in $\text{Spec } Z_{0,\mathbf{h}}$ are in a natural bijection with l -multipartitions of n . On the other hand, the \mathbb{C}^* -fixed points in $\text{Hilb}(K)$ can be described as monomial ideals in $\mathbb{C}[x, y]$ of colength K and are therefore classified by partitions of K .

The fixed points lying in the image of (1) have the same l -core ν (depending on \mathbf{h}). The map (1) therefore induces a bijection $\mathcal{P}(l, n) \longleftrightarrow \mathcal{P}_\nu(K)$. Claim A can be reduced to the following statement about \mathbb{C}^* -fixed points.

Claim B. *The bijection $\mathcal{P}(l, n) \longleftrightarrow \mathcal{P}_\nu(K)$ induced by (1) is given by the \mathbf{h} -twisted l -quotient bijection.*

Claim B has an interesting application to combinatorics. We will consider certain vector bundles on $\text{Spec } Z_{0,\mathbf{h}}$ and the corresponding quiver variety $\mathcal{X}_{\theta_\mathbf{h}}(n\delta)$. By comparing the Poincaré polynomials

of their fibres at the \mathbb{C}^* -fixed points we will obtain combinatorial identities which generalize the q -hook formula.

1.4 The proof. To prove Claim B we split the problem into two parts and consider the correspondences between the fixed points induced by the Etingof-Ginzburg map and the reflection functors separately. Our first step is to explicitly construct the \mathbb{C}^* -fixed points in $\mathcal{X}_{\theta_h}(n\delta)$ as isomorphism classes of certain quiver representations. We show that these fixed points are in a natural bijection with $\mathcal{P}_{\varnothing}(nl)$, the set of partitions of nl with a trivial l -core.

Our next step is to identify the bijection $\mathcal{P}(l, n) \rightarrow \mathcal{P}_{\varnothing}(nl)$ induced by the Etingof-Ginzburg isomorphism. We show that the inverse of this bijection sends a partition μ to the reverse of the quotient of μ (see §2.2 for the terminology). The proof of this fact is rather involved. Using the representation theory of degenerate affine Hecke algebras we first construct the following commutative diagram.

$$\begin{array}{ccc} \text{Spec } Z_{0,h} & \xrightarrow{\text{EG}} & \mathcal{X}_{\theta_h}(n\delta) \\ & \searrow \rho_1 \quad \swarrow \rho_2 & \\ & \mathbb{C}^n / S_n & \end{array} \quad (2)$$

We then determine the images of the \mathbb{C}^* -fixed points under the maps ρ_1 and ρ_2 . The map ρ_1 simply sends a fixed point labelled by multipartition $\underline{\lambda}$ to the residue of $\underline{\lambda}$ while the map ρ_2 sends a quiver representation to a certain subset of its eigenvalues. Given a fixed point x labelled by a partition μ , we obtain an explicit formula for $\rho_2(x)$ in terms of the Frobenius form of μ . We then show that this formula defines the residue of the reverse of the l -quotient of μ . Our argument at this point becomes purely combinatorial. It mainly relies on the combinatorics of abaci (bead diagrams) and some inductive techniques. The following theorem summarizes our main results so far.

Theorem A. (i) The closed \mathbb{C}^* -fixed points in \mathcal{X}_{θ_h} are in a natural bijection with the set $\mathcal{P}_{\varnothing}(nl)$.
(ii) The map $\text{Spec } Z_{0,h} \xrightarrow{\text{EG}} \mathcal{X}_{\theta_h}$ induces a bijection

$$\mathcal{P}(l, n) \rightarrow \mathcal{P}_{\varnothing}(nl), \quad (\text{Quot}(\mu))^{\flat} \mapsto \mu.$$

Here $(\text{Quot}(\mu))^{\flat}$ denotes the reverse of the l -quotient of μ (see §2.2 for the terminology).

We subsequently consider the correspondences between the \mathbb{C}^* -fixed points induced by reflection functors. Reflection functors define morphisms $\mathfrak{R}_i : \mathcal{X}_{\theta}(\mathbf{d}) \rightarrow \mathcal{X}_{\theta'}(\mathbf{d}')$ between quiver varieties associated to different dimension vectors and stability conditions. We can restrict attention to dimension vectors of the form $\mathbf{d} = n\delta + \mathbf{d}_{\nu}$, where \mathbf{d}_{ν} is a dimension vector corresponding to some l -core ν . Reflection functors induce bijections between the labelling sets of the \mathbb{C}^* -fixed points

$$\mathbf{R}_i : \mathcal{P}_{\nu}(nl + |\nu|) \rightarrow \mathcal{P}_{\nu'}(nl + |\nu'|),$$

where ν and ν' are possibly different l -cores. Van Leeuwen defined in [18] an action of the affine symmetric group $\tilde{S}_l = \langle \sigma_0, \dots, \sigma_{l-1} \rangle$ on the set of all partitions. We prove the following result.

Theorem B. Let $\mu \in \mathcal{P}_{\nu}(nl + |\nu|)$. Then

$$\mathbf{R}_i(\mu) = (\sigma_i * \mu^t)^t,$$

i.e., the action of reflection functors on the \mathbb{C}^* -fixed points coincides (up to transposes) with the \tilde{S}_l -action.

We also explicitly determine the l -core and l -quotient of the partition $\mathbf{R}_i(\mu)$. Using Theorem A and repeatedly applying Theorem B we are able to deduce Claim B.

1.5 The cyclotomic q -hook formula. We are now going to discuss an application of our results to combinatorics - a new proof and a generalization of the q -hook formula. Let us first recall what the q -hook formula states. We must introduce some notation. Let μ be a partition of n . By $\square \in \mu$ we mean a cell in the Young diagram of μ and by $c(\square)$ we mean the content of that cell. Let $f_\mu(t)$ denote the fake degree polynomial associated to μ . The q -hook formula states that

$$\sum_{\square \in \mu} t^{c(\square)} = [n]_t \sum_{\lambda \uparrow \mu} \frac{f_\lambda(t)}{f_\mu(t)}, \quad (3)$$

where the sum on the RHS ranges over subpartitions of μ obtained by deleting precisely one cell in the Young diagram of μ . The q -hook formula has been proven using probabilistic, combinatorial and algebraic methods ([15], [2], [8]).

In our proof of the q -hook formula we use certain vector bundles. Let e_n denote the symmetrizing idempotent in Γ_n . The right $e_n \mathbb{H}_{0, \mathbf{h}} e_n$ -module $\mathbb{H}_{0, \mathbf{h}} e_n$ defines a coherent sheaf on $\text{Spec } Z_{0, \mathbf{h}}$. Since we are assuming that the variety $\text{Spec } Z_{0, \mathbf{h}}$ is smooth, this sheaf is also locally free. Let \mathcal{R} denote the total space of the corresponding vector bundle. Each fibre carries an action of the group Γ_n . Set $\Gamma_{n-1} := (\mathbb{Z}/l\mathbb{Z}) \wr S_{n-1}$ and let $\mathcal{R}^{\Gamma_{n-1}}$ denote the subbundle of \mathcal{R} consisting of Γ_{n-1} -invariants.

Now consider the principal $G(\mathbf{d})$ -bundle $\mu^{-1}(\theta_{\mathbf{h}}) \rightarrow \mathcal{X}_{\theta_{\mathbf{h}}}(n\delta)$ (see §8.1 for the notation). The trivial vector bundle $\mu^{-1}(\theta_{\mathbf{h}}) \times \mathbb{C}^{nl}$ descends to the vector bundle $\mu^{-1}(\theta_{\mathbf{h}}) \times^{G(\mathbf{d})} \mathbb{C}^{nl} \rightarrow \mathcal{X}_{\theta_{\mathbf{h}}}(n\delta)$. We call it the tautological bundle on $\mathcal{X}_{\theta_{\mathbf{h}}}$ and denote its total space by \mathcal{V} . Etingof and Ginzburg showed in [7, §11] that there exists an isomorphism of vector bundles

$$\begin{array}{ccc} \mathcal{R}^{\Gamma_{n-1}} & \xrightarrow{\sim} & \mathcal{V} \\ \downarrow & & \downarrow \\ \text{Spec } Z_{0, \mathbf{h}} & \xrightarrow{\sim} & \mathcal{X}_{\theta_{\mathbf{h}}}(n\delta) \end{array}$$

lifting the Etingof-Ginzburg map. We calculate the \mathbb{C}^* -characters of the fibres of these bundles at the fixed points. More specifically, we show the following (see §6.3 and §7.3 for a detailed explanation of the notation).

Theorem C. *Let $\mu \in \mathcal{P}_{\emptyset}(nl)$ and $\underline{\gamma} \in \mathcal{P}(l, n)$. The \mathbb{C}^* -characters of the fibres \mathcal{V}_μ and $(\mathcal{R}^{\Gamma_{n-1}})_{\underline{\gamma}}$ are given by the following formulas:*

$$\text{ch}_t \mathcal{V}_\mu = \sum_{\square \in \mu} t^{c(\square)}, \quad \text{ch}_t (\mathcal{R}^{\Gamma_{n-1}})_{\underline{\gamma}} = [nl]_t \sum_{\underline{\lambda} \uparrow \underline{\gamma}} \frac{f_{\underline{\lambda}}(t)}{f_{\underline{\gamma}}(t)}.$$

We now use Theorem A to relate these two characters. We obtain the following identity expressing the residue of a partition in terms of fake degree polynomials.

Theorem D. *Let $\mu \in \mathcal{P}_{\emptyset}(nl)$ and let $\underline{\text{Quot}}(\mu)$ be the l -quotient of μ . Then:*

$$\sum_{\square \in \mu} t^{c(\square)} = [nl]_t \sum_{\underline{\lambda} \uparrow (\underline{\text{Quot}}(\mu))^{\flat}} \frac{f_{\underline{\lambda}}(t)}{f_{(\underline{\text{Quot}}(\mu))^{\flat}}(t)}. \quad (4)$$

We call (4) the *cyclotomic q -hook formula*. If we set $l = 1$ we recover the classical q -hook formula.

1.6 Structure of the paper. The paper is divided into three parts. The first part is devoted to rational Cherednik algebras, the second part to quiver varieties and Hilbert schemes, while the third part establishes the correspondence between the \mathbb{C}^* -fixed points. Let us briefly summarize the contents of each section.

In section 2 we review basic facts about the representation theory of generalized symmetric groups. In section 3 we recall the definition of the rational Cherednik algebra $\mathbb{H}_{0, \mathbf{h}}$ and review the

basic properties of the spectrum of its centre. We define a \mathbb{C}^* -action on $\text{Spec } Z_{0,\mathbf{h}}$. In section 4 we identify the \mathbb{C}^* -fixed points as the annihilators of the simple quotients of baby Verma modules. In section 5 we provide several equivalent characterizations of the tautological vector bundle on $\text{Spec } Z_{0,\mathbf{h}}$ as well as its subbundle consisting of Γ_{n-1} -invariants. In section 6 we introduce notations related to the combinatorics of Young tableaux. Section 7 is dedicated to the calculation of the characters of the fibres of the vector bundle $\mathcal{R}^{\Gamma_{n-1}}$. We thus establish the second formula in Theorem C.

We subsequently proceed to discuss quiver varieties. In section 8 we define the varieties $\mathcal{X}_\theta(\mathbf{d})$ and $\mathcal{M}_\theta(\mathbf{d})$ as well as recall the construction of the Etingof-Ginzburg map. Section 9 is devoted to the combinatorics of abaci. In section 10 we recall the definition of reflection functors and explain how $\text{Spec } Z_{0,\mathbf{h}}$ can be (non-algebraically) embedded into a Hilbert scheme. In section 11 we pose the problem of matching the \mathbb{C}^* -fixed points and present counterexamples to Gordon's proof.

In section 12 we construct the \mathbb{C}^* -fixed points in the quiver varieties $\mathcal{X}_\theta(\mathbf{d})$. We prove the first part of Theorem A as well as the first formula in Theorem C. In section 13 we use the representation theory of degenerate affine Hecke algebras to construct the commutative diagram (2). We also calculate the images of the \mathbb{C}^* -fixed points under maps ρ_1 and ρ_2 . In section 14 we prove the second part of Theorem A as well as Theorem D. Section 15 contains the proof of Theorem B.

1.7 Conventions. In this paper we consider smooth quasi-projective varieties which also have a hyper-Kähler structure. As such we will consider them both as schemes and hyper-Kähler manifolds, depending on the context. If R is a ring, by $\text{Spec } R$ we either mean its prime spectrum or maximal spectrum, again depending on the context (often both interpretations are correct). When we wish to emphasize that we are talking about the maximal rather than prime spectrum, we use the notation $\text{MaxSpec } R$.

We will also encounter another notational problem. A lot of the symbols we use contain integral indices. Some of these indices should be considered modulo l while others shouldn't. Whenever we introduce a problematic symbol, we will indicate which group it belongs to.

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Part I: Rational Cherednik algebras

2 Generalized symmetric groups

In this section we recall some facts and notations concerning the generalized symmetric group $(\mathbb{Z}/l\mathbb{Z}) \wr S_n$ and its representation theory.

2.1 Generalized symmetric groups. Let us fix once and for all two positive integers n, l . We regard the symmetric group S_n as the group of permutations of the set $\{1, \dots, n\}$. For $1 \leq i < j \leq n$ let $s_{i,j}$ denote the simple transposition swapping numbers i and j and leaving all the other numbers fixed. We abbreviate $s_i = s_{i,i+1}$ for $i = 1, \dots, l-1$ and $s_0 = s_n = s_{1,n}$. Let us fix a finite cyclic group $C_l = \mathbb{Z}/l\mathbb{Z} = \{1, \epsilon, \epsilon^2, \dots, \epsilon^{l-1}\}$ and set $\Gamma_n = C_l \wr S_n = (C_l)^n \rtimes S_n$, the wreath product of C_l and S_n . It is a complex reflection group of type $G(l, 1, n)$. For $1 \leq i \leq n$ and $1 \leq j \leq l-1$ let ϵ_i^j denote the element $(1, \dots, 1, \epsilon^j, 1, \dots, 1) \in (C_l)^n$ which is non-trivial only in the i -th coordinate.

We regard S_{n-1} as the subgroup of S_n generated by the simple transpositions $s_{2,3}, \dots, s_{n-1,n}$. We also regard $(C_l)^{n-1}$ as a subgroup of $(C_l)^n$ consisting of elements whose first coordinate is

equal to one. This determines an embedding $\Gamma_{n-1} \hookrightarrow \Gamma_n$. Let

$$e_{n-1} = \frac{1}{l^{n-1}(n-1)!} \sum_{g \in \Gamma_{n-1}} g, \quad e_n = \frac{1}{l^n n!} \sum_{g \in \Gamma_n} g$$

be the corresponding symmetrizing idempotents. We have $(\mathbb{C}\Gamma_n)^{\Gamma_{n-1}} = e_{n-1}\mathbb{C}\Gamma_n$. Note that $|(\mathbb{C}\Gamma_n)^{\Gamma_{n-1}}| = nl$.

2.2 Partitions and multipartitions. Let k be a non-negative integer. A partition λ of k is an infinite non-increasing sequence $(\lambda_1, \lambda_2, \lambda_3, \dots)$ of non-negative integers such that $\sum_{i=1}^{\infty} \lambda_i = k$. We write $|\lambda| = k$ and denote the set of all partitions of k by $\mathcal{P}(k)$. We say that $\mu = (\mu_1, \mu_2, \mu_3, \dots)$ is a subpartition of λ if μ is a partition of some positive integer $m \leq k$ and $\mu_i \leq \lambda_i$ for all $i = 1, 2, \dots$. A subpartition μ of λ is called a restriction of λ , denoted $\mu \uparrow \lambda$, if $|\mu| = k - 1$.

An l -composition α of k is an l -tuple $\alpha_0, \dots, \alpha_{l-1}$ of non-negative integers such that $\sum_{i=0}^{l-1} \alpha_i = k$. An l -multipartition $\underline{\lambda}$ of k is an l -tuple $(\lambda^0, \dots, \lambda^{l-1})$ such that each λ^i is a partition and $\sum_{i=0}^{l-1} |\lambda^i| = k$. We consider the upper indices modulo l . Let $\mathcal{P}(l, k)$ denote the set of l -multipartitions of k . We say that $\underline{\mu} = (\mu^0, \dots, \mu^{l-1})$ is a submultipartition of $\underline{\lambda}$ if μ^i is a subpartition of λ^i , for each $i = 0, \dots, l-1$. We call a submultipartition $\underline{\mu}$ of $\underline{\lambda}$ a restriction of $\underline{\lambda}$, denoted $\underline{\mu} \uparrow \underline{\lambda}$, if $\sum_{i=0}^{l-1} |\mu^i| = k - 1$.

If λ is a partition we denote its transpose by λ^t . If $\underline{\lambda} = (\lambda^0, \dots, \lambda^{l-1}) \in \mathcal{P}(l, k)$, we call $\underline{\lambda}^t = ((\lambda^0)^t, \dots, (\lambda^{l-1})^t)$ the *transpose* multipartition and $\underline{\lambda}^b := (\lambda^{l-1}, \lambda^{l-2}, \dots, \lambda^0)$ the *reverse* multipartition. Finally, we set

$$\mathcal{P} = \bigsqcup_{k \in \mathbb{Z}_{\geq 0}} \mathcal{P}(k), \quad \mathcal{P} = \bigsqcup_{k \in \mathbb{Z}_{\geq 0}} \mathcal{P}(l, k).$$

2.3 Representations of Γ_n . Let $\eta := e^{2\pi i/l}$ be an l -th primitive root of unity. For $j = 0, \dots, l-1$ let M_j be the (unique up to isomorphism) irreducible C_l -module such that $\epsilon \in C_l$ acts on M_k by the scalar η^j . Let $\underline{\lambda} = (\lambda^0, \dots, \lambda^{l-1})$ be an l -multipartition of n and let $\alpha = (|\lambda^0|, \dots, |\lambda^{l-1}|)$ be the corresponding l -composition of n . We define $E(\alpha) := M_0^{\otimes |\lambda^0|} \otimes \dots \otimes M_{l-1}^{\otimes |\lambda^{l-1}|}$. The vector space $E(\alpha)$ carries a natural structure of a $(C_l)^n$ -module. If $g = (g_1, \dots, g_n) \in (C_l)^n$ and $(v_1 \otimes \dots \otimes v_n) \in E(\alpha)$, then $g \cdot (v_1 \otimes \dots \otimes v_n) = (g_1.v_1 \otimes \dots \otimes g_n.v_n)$. The inertia subgroup of $E(\alpha)$ in Γ_n is $C_l \wr S_\alpha$, where $S_\alpha = S_{|\lambda^0|} \times \dots \times S_{|\lambda^{l-1}|}$ is a Young subgroup of S_n . We can make $E(\alpha)$ into a $C_l \wr S_\alpha$ -module by setting $(g, \sigma) \cdot (v_1 \otimes \dots \otimes v_n) = (g_1.v_{\sigma^{-1}(1)} \otimes \dots \otimes g_n.v_{\sigma^{-1}(n)})$.

For each partition λ^i let $D(\lambda^i)$ be the Specht module of $S_{|\lambda^i|}$ corresponding to the partition λ^i . Then $D(\underline{\lambda}) := D(\lambda^0) \otimes \dots \otimes D(\lambda^{l-1})$ is naturally a module over S_α . By letting $(C_l)^n$ act trivially on $D(\underline{\lambda})$ we can regard it as a module over $C_l \wr S_\alpha$. We finally define

$$C(\underline{\lambda}) := \text{Ind}_{C_l \wr S_\alpha}^{C_l \wr S_n} E(\alpha) \otimes D(\underline{\lambda}).$$

We call it the Specht module associated to the l -multipartition $\underline{\lambda}$. Let $C(\underline{\lambda})^*$ denote the dual of $C(\underline{\lambda})$. We use the notation triv for the trivial representation of Γ_n .

Proposition 2.1. *The modules $\{C(\underline{\lambda}) \mid \underline{\lambda} \in \mathcal{P}(l, n)\}$ form a complete and irredundant set of isomorphism classes of irreducible Γ_n -modules. Moreover, for each $\underline{\lambda} \in \mathcal{P}(l, n)$ we have the following branching rule*

$$C(\underline{\lambda})|_{\Gamma_{n-1}} := \text{Res}_{\Gamma_{n-1}}^{\Gamma_n} C(\underline{\lambda}) = \bigoplus_{\underline{\mu} \uparrow \underline{\lambda}} C(\underline{\mu}).$$

Proof. The first statement is a corollary of [13, Theorem 4.4.3]. The branching rule follows from [24, Theorem 10]. \square

3 Rational Cherednik algebras and their centres

We are now going to recall the definition of the rational Cherednik algebra $\mathbb{H}_{0,\mathbf{h}}$ associated to the complex reflection group Γ_n . We will review some basic facts about its centre $Z_{0,\mathbf{h}}$ and the affine variety $\text{Spec } Z_{0,\mathbf{h}}$. We will also define a \mathbb{C}^* -action on $\text{Spec } Z_{0,\mathbf{h}}$.

3.1 Rational Cherednik algebras. Let \mathfrak{h} be the reflection representation of Γ_n and \mathfrak{h}^* its dual. Let us choose a basis x_1, \dots, x_n of \mathfrak{h}^* and a dual basis y_1, \dots, y_n of \mathfrak{h} such that $\epsilon_i \sigma \cdot y_j = \eta^{-\delta_{i,\sigma(j)}} y_{\sigma(j)}$ and $\epsilon_i \sigma \cdot x_j = \eta^{\delta_{i,\sigma(j)}} x_{\sigma(j)}$. We define a symplectic form ω on $\mathfrak{h} \oplus \mathfrak{h}^*$ by setting $\omega((y, x), (y', x')) = x'(y) - x(y')$, for $x, x' \in \mathfrak{h}^*$ and $y, y' \in \mathfrak{h}$.

Definition 3.1. Let us choose a parameter $\mathbf{h} = (h, H_1, \dots, H_{l-1}) \in \mathbb{Q}^l$ and set $H_0 = -(H_1 + \dots + H_{l-1})$. The *rational Cherednik algebra* $\mathbb{H}_{0,\mathbf{h}}$ associated to Γ_n is the quotient of the cross-product $T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes \mathbb{C}\Gamma_n$ by the relations

- $[x_i, x_j] = [y_i, y_j] = 0$ for all $1 \leq i, j \leq n$,
- $[x_i, y_j] = -h \sum_{k=0}^{l-1} \eta^k s_{i,j} \epsilon_i^k \epsilon_j^{-k}$ for all $1 \leq i \neq j \leq n$,
- $[x_i, y_i] = h \sum_{j \neq i} \sum_{k=0}^{l-1} s_{i,j} \epsilon_i^k \epsilon_j^{-k} + \sum_{k=0}^{l-1} (\sum_{m=0}^{l-1} \eta^{-mk} H_m) \epsilon_i^k$ for all $1 \leq i \leq n$.

Remark 3.2. Setting

$$c = h, \quad c_k = \sum_{m=0}^{l-1} \eta^{-mk} H_m \quad (k = 0, \dots, l-1) \quad (5)$$

we obtain another parametrization of $H_{0,\mathbf{h}}$.

3.2 The centre of $\mathbb{H}_{0,\mathbf{h}}$. We let $Z_{0,\mathbf{h}}$ denote the centre of $\mathbb{H}_{0,\mathbf{h}}$. It can be related to the *spherical rational Cherednik algebra* $e_n \mathbb{H}_{0,\mathbf{h}} e_n$ through the Satake isomorphism.

Theorem 3.3. *The map $Z_{0,\mathbf{h}} \rightarrow e_n \mathbb{H}_{0,\mathbf{h}} e_n$, $z \mapsto z \cdot e_n$ is an algebra isomorphism.*

Proof. See [7, Theorem 3.1]. □

3.3 The \mathbb{C}^* -action on $\mathbb{H}_{0,\mathbf{h}}$ and $Z_{0,\mathbf{h}}$. Let $t \in \mathbb{C}^*$. We define the \mathbb{C}^* -action on $\mathbb{H}_{0,\mathbf{h}}$ by the rule $t \cdot x_i = t x_i$, $t \cdot y_i = t^{-1} y_i$, $t \cdot g = g$, for $1 \leq i \leq n$ and $g \in \Gamma_n$. We can see from the relations in $\mathbb{H}_{0,\mathbf{h}}$ that this action is well-defined. The \mathbb{C}^* -action on $H_{0,\mathbf{h}}$ restricts to an action on the spherical subalgebra $e_n \mathbb{H}_{0,\mathbf{h}} e_n$. Since the Satake isomorphism $Z_{0,\mathbf{h}} \rightarrow e_n \mathbb{H}_{0,\mathbf{h}} e_n$ is \mathbb{C}^* -equivariant, we obtain a \mathbb{C}^* -action on $Z_{0,\mathbf{h}}$.

Note that the \mathbb{C}^* -action defines a \mathbb{Z} -grading on $\mathbb{H}_{0,\mathbf{h}}$ with $\deg x_i = 1$, $\deg y_i = -1$ and $\deg g = 0$ for $1 \leq i \leq n$ and $g \in \Gamma_n$. The homogeneous elements of degree m are precisely those elements on which \mathbb{C}^* acts with weight m .

Notation. Let V be a \mathbb{Z} -graded vector space. Let $\text{ch}_t V \in \mathbb{Z}[[t, t^{-1}]]$ denote its Poincaré series. Since a \mathbb{Z} -grading is equivalent to a \mathbb{C}^* -module structure, we can regard $\text{ch}_t V$ as the \mathbb{C}^* -character of V .

3.4 The variety $\text{Spec } Z_{0,\mathbf{h}}$. The variety $\text{Spec } Z_{0,\mathbf{h}}$ can be regarded as a moduli space of irreducible $\mathbb{H}_{0,\mathbf{h}}$ -modules. Let $\text{Irrep}(\mathbb{H}_{0,\mathbf{h}})$ denote the set of all irreducible representations of $\mathbb{H}_{0,\mathbf{h}}$. If $M \in \text{Irrep}(\mathbb{H}_{0,\mathbf{h}})$, let $\chi_M : Z_{0,\mathbf{h}} \rightarrow \mathbb{C}$ denote the character by which $Z_{0,\mathbf{h}}$ acts on M . It is a consequence of the Artin-Procesi theorem that $\chi = \chi_M$ for a unique, up to isomorphism, irreducible $\mathbb{H}_{0,\mathbf{h}}$ -module M if and only if the maximal ideal $\ker \chi$ lies in the Azumaya locus of $\mathbb{H}_{0,\mathbf{h}}$ over $Z_{0,\mathbf{h}}$. In this case the module M is isomorphic to the regular representation $\mathbb{C}\Gamma_n$ as a $\mathbb{C}\Gamma_n$ -module. Furthermore, the Azumaya locus of $\mathbb{H}_{0,\mathbf{h}}$ over $Z_{0,\mathbf{h}}$ coincides with the smooth locus of $\text{Spec } Z_{0,\mathbf{h}}$. The following proposition gives a necessary and sufficient condition for the variety $\text{Spec } Z_{0,\mathbf{h}}$ to be smooth.

Proposition 3.4. Let $\mathbf{h} = (h, H_1, \dots, H_{l-1}) \in \mathbb{Q}^l$. The variety $\text{Spec } Z_{0,\mathbf{h}}$ is singular if and only if

$$(H_i + \dots + H_j) + mh = 0 \quad \text{or} \quad h = 0,$$

for some $1 \leq i \leq j \leq l-1$ and $1-n \leq m \leq n-1$.

Proof. See [10, Lemma 4.3]. \square

From now on we will always assume that the parameter \mathbf{h} is chosen so that the variety $\text{Spec } Z_{0,\mathbf{h}}$ is smooth. We thus have a bijection

$$\text{Irrep}(\mathbb{H}_{0,\mathbf{h}})/\sim \longleftrightarrow \text{MaxSpec } Z_{0,\mathbf{h}}, \quad M \mapsto \ker \chi_M, \quad M_\chi \mapsto \mathfrak{m} = \ker \chi. \quad (6)$$

Proposition 3.5. The variety $\text{Spec } Z_{0,\mathbf{h}}$ is irreducible.

Proof. By [7, Theorem 3.3] one can introduce a filtration on $Z_{0,\mathbf{h}}$ so that

$$\text{gr } Z_{0,\mathbf{h}} \cong \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]^{\Gamma_n}.$$

Hence $Z_{0,\mathbf{h}}$ is a finitely generated integral domain and its spectrum is an irreducible affine algebraic variety. \square

3.5 The variety $\text{Rep}_{\mathbb{C}\Gamma_n}(\mathbb{H}_{0,\mathbf{h}})$. We are now going to recall that $\text{Spec } Z_{0,\mathbf{h}}$ can be regarded as a quotient of the variety $\text{Rep}_{\mathbb{C}\Gamma_n}^o(\mathbb{H}_{0,\mathbf{h}})$. This fact will be important when we consider the tautological bundle on $\text{Spec } Z_{0,\mathbf{h}}$.

Definition 3.6. Let $\text{Rep}_{\mathbb{C}\Gamma_n}(\mathbb{H}_{0,\mathbf{h}})$ be the variety of all algebra homomorphisms $\mathbb{H}_{0,\mathbf{h}} \rightarrow \text{End}_{\mathbb{C}}(\mathbb{C}\Gamma_n)$ whose restriction to $\mathbb{C}\Gamma_n \subset \mathbb{H}_{0,\mathbf{h}}$ is the $\mathbb{C}\Gamma_n$ -action by left multiplication, i.e., the regular representation. This is an affine algebraic variety.

Let $\phi \in \text{Rep}_{\mathbb{C}\Gamma_n}(\mathbb{H}_{0,\mathbf{h}})$. The one-dimensional vector space $e_n \mathbb{C}\Gamma_n$ is stable under all the endomorphisms in $\phi(e_n \mathbb{H}_{0,\mathbf{h}} e_n)$. Therefore ϕ restricts to an algebra homomorphism $\chi_\phi : Z_{0,\mathbf{h}} \cong e_n \mathbb{H}_{0,\mathbf{h}} e_n \rightarrow \text{End}_{\mathbb{C}}(e_n \mathbb{C}\Gamma_n) \cong \mathbb{C}$. We obtain a morphism of algebraic varieties

$$\pi : \text{Rep}_{\mathbb{C}\Gamma_n}(\mathbb{H}_{0,\mathbf{h}}) \rightarrow \text{Spec } Z_{0,\mathbf{h}}, \quad \phi \mapsto \ker \chi_\phi. \quad (7)$$

Definition 3.7. Let $\text{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n)$ be the group of \mathbb{C} -linear Γ_n -equivariant automorphisms of $\mathbb{C}\Gamma_n$.

We have an isomorphism $\text{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n) \cong \prod_{\lambda \in \mathcal{P}(l,n)} \text{GL}(d(\lambda), \mathbb{C})$, where $d(\lambda) = \dim_{\mathbb{C}} C(\lambda)$. The group $\text{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n)$ acts naturally on $\text{Rep}_{\mathbb{C}\Gamma_n}(\mathbb{H}_{0,\mathbf{h}})$: if $g \in \text{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n)$ and $\phi \in \text{Rep}_{\mathbb{C}\Gamma_n}(\mathbb{H}_{0,\mathbf{h}})$ then $(g.\phi)(z) = g\phi(z)g^{-1}$, for all $z \in \mathbb{H}_{0,\mathbf{h}}$. Moreover, each fibre of the map π is stable under the action of $\text{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n)$.

Theorem 3.8. There exists an irreducible component $\text{Rep}_{\mathbb{C}\Gamma_n}^o(\mathbb{H}_{0,\mathbf{h}})$ of $\text{Rep}_{\mathbb{C}\Gamma_n}(\mathbb{H}_{0,\mathbf{h}})$ such that the map (7) induces an isomorphism of algebraic varieties

$$\tilde{\pi} : \text{Rep}_{\mathbb{C}\Gamma_n}^o(\mathbb{H}_{0,\mathbf{h}}) // \text{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n) = \text{Spec } \mathbb{C}[\text{Rep}_{\mathbb{C}\Gamma_n}^o(\mathbb{H}_{0,\mathbf{h}})]^{\text{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n)} \rightarrow \text{Spec } Z_{0,\mathbf{h}}. \quad (8)$$

Proof. See [7, Theorem 3.7]. \square

3.6 The \mathbb{C}^* -action on $\text{Rep}_{\mathbb{C}\Gamma_n}(\mathbb{H}_{0,\mathbf{h}})$ and $\text{Spec } Z_{0,\mathbf{h}}$. The \mathbb{C}^* -action on $\mathbb{H}_{0,\mathbf{h}}$ induces \mathbb{C}^* -actions on the varieties $\text{Rep}_{\mathbb{C}\Gamma_n}(\mathbb{H}_{0,\mathbf{h}})$ and $\text{Spec } Z_{0,\mathbf{h}}$. These actions can be described explicitly in the following way.

Definition 3.9. Let $t \in \mathbb{C}^*$ and $\phi \in \text{Rep}_{\mathbb{C}\Gamma_n}(\mathbb{H}_{0,\mathbf{h}})$. Set $(t.\phi)(z) := \phi(t^{-1}.z)$ for all $z \in \mathbb{H}_{0,\mathbf{h}}$. If \mathfrak{m} is a closed point of $\text{Spec } Z_{0,\mathbf{h}}$, i.e., a maximal ideal in $Z_{0,\mathbf{h}}$, then set $t.\mathfrak{m} := \{t.z \mid z \in \mathfrak{m}\}$.

Lemma 3.10. The map π is \mathbb{C}^* -equivariant.

Proof. Let $\mathfrak{m} = \ker \chi_\phi = \pi(\phi)$. We have $\phi(\mathfrak{m})(e_n \mathbb{C}\Gamma_n) = \{0\}$ and

$$(t.\phi)(t.\mathfrak{m})(e_n \mathbb{C}\Gamma_n) = \phi(tt^{-1}.\mathfrak{m})(e_n \mathbb{C}\Gamma_n) = \phi(\mathfrak{m})(e_n \mathbb{C}\Gamma_n) = \{0\}.$$

It follows that the endomorphisms in $(t.\phi)(t.\mathfrak{m})$ annihilate $e_n \mathbb{C}\Gamma_n$ and so $t.\mathfrak{m} = \ker \chi_{t.\phi}$. \square

Lemma 3.10 implies that the isomorphism $\tilde{\pi}$ is also \mathbb{C}^* -equivariant.

4 The \mathbb{C}^* -fixed points in $\text{Spec } Z_{0,\mathfrak{h}}$

The goal of this section is to explain how the \mathbb{C}^* -fixed points in $\text{MaxSpec } Z_{0,\mathfrak{h}}$ are classified by l -multipartitions of n . We recall the definitions of restricted rational Cherednik algebras and baby Verma modules. The annihilators of the simple quotients of the latter are precisely the \mathbb{C}^* -fixed points. All the results in this section were proven by Gordon in [9], but we include the (rather short and elegant) proofs for the reader's convenience.

4.1 Restricted rational Cherednik algebras. The subalgebra $\mathbb{C}[\mathfrak{h}]^{\Gamma_n} \otimes \mathbb{C}[\mathfrak{h}^*]^{\Gamma_n}$ of $\mathbb{H}_{0,\mathfrak{h}}$ is contained in $Z_{0,\mathfrak{h}}$ and $Z_{0,\mathfrak{h}}$ is a free $\mathbb{C}[\mathfrak{h}]^{\Gamma_n} \otimes \mathbb{C}[\mathfrak{h}^*]^{\Gamma_n}$ -module of rank $|\Gamma_n|$. The inclusion $\mathbb{C}[\mathfrak{h}]^{\Gamma_n} \otimes \mathbb{C}[\mathfrak{h}^*]^{\Gamma_n} \hookrightarrow Z_{0,\mathfrak{h}}$ induces a \mathbb{C}^* -equivariant morphism of algebraic varieties

$$\Upsilon : \text{Spec } Z_{0,\mathfrak{h}} \rightarrow \mathfrak{h}/\Gamma_n \times \mathfrak{h}^*/\Gamma_n.$$

Lemma 4.1. *We have*

$$(\text{Spec } Z_{0,\mathfrak{h}})^{\mathbb{C}^*} = \Upsilon^{-1}(0).$$

Proof. The \mathbb{C}^* -action on $\mathfrak{h}/\Gamma_n \times \mathfrak{h}^*/\Gamma_n$ is induced by the \mathbb{C}^* -action on $\mathfrak{h} \times \mathfrak{h}^*$. The latter is given by $t.(y, x) = (t^{-1}y, tx)$, so the only \mathbb{C}^* -fixed point in $\mathfrak{h}/\Gamma_n \times \mathfrak{h}^*/\Gamma_n$ is 0. Therefore $(\text{Spec } Z_{0,\mathfrak{h}})^{\mathbb{C}^*} \subseteq \Upsilon^{-1}(0)$. Let $p \in \Upsilon^{-1}(0)$ and consider the orbit map $\mathbb{C}^* \rightarrow \Upsilon^{-1}(0)$, $t \mapsto t.p$. The group \mathbb{C}^* is connected and the fibre $\Upsilon^{-1}(0)$ is finite so the image of the orbit map must consist of the single point p . It follows that $(\text{Spec } Z_{0,\mathfrak{h}})^{\mathbb{C}^*} = \Upsilon^{-1}(0)$. \square

We are now going to identify the \mathbb{C}^* -fixed points in $\text{MaxSpec } Z_{0,\mathfrak{h}}$ with isomorphism classes of irreducible $\mathbb{H}_{0,\mathfrak{h}}$ -modules under the bijection (6).

Definition 4.2. Let $\mathbb{C}[\mathfrak{h}]_+^{\Gamma_n}$ (resp. $\mathbb{C}[\mathfrak{h}^*]_-^{\Gamma_n}$) denote the ideal of $\mathbb{C}[\mathfrak{h}]^{\Gamma_n}$ (resp. $\mathbb{C}[\mathfrak{h}^*]^{\Gamma_n}$) generated by homogeneous elements of positive (resp. negative) degree, in the grading defined by the \mathbb{C}^* -action on $\mathbb{H}_{0,\mathfrak{h}}$. We call

$$\overline{\mathbb{H}}_{0,\mathfrak{h}} := \mathbb{H}_{0,\mathfrak{h}} / \mathbb{H}_{0,\mathfrak{h}} \cdot (\mathbb{C}[\mathfrak{h}]_+^{\Gamma_n} \otimes \mathbb{C}[\mathfrak{h}^*]_-^{\Gamma_n})$$

the *restricted rational Cherednik algebra*. It is a finite-dimensional algebra.

Lemma 4.3. *There is a bijection between the closed points of $\Upsilon^{-1}(0)$ and isomorphism classes of simple modules over the restricted rational Cherednik algebra $\overline{\mathbb{H}}_{0,\mathfrak{h}}$.*

Proof. The closed points of $\Upsilon^{-1}(0)$ can be characterized as those maximal ideals \mathfrak{m} of $Z_{0,\mathfrak{h}}$ with the property that $\mathfrak{m} \cap (\mathbb{C}[\mathfrak{h}]_+^{\Gamma_n} \otimes \mathbb{C}[\mathfrak{h}^*]_-^{\Gamma_n}) = \mathbb{C}[\mathfrak{h}]_+^{\Gamma_n} \otimes \mathbb{C}[\mathfrak{h}^*]_-^{\Gamma_n}$. Suppose that $\mathfrak{m} \in \Upsilon^{-1}(0)$. Since $\text{Spec } Z_{0,\mathfrak{h}}$ is smooth we have $\mathfrak{m} = \ker \chi_M$ for a unique, up to isomorphism, simple module $M \in \text{Irrep}(\mathbb{H}_{0,\mathfrak{h}})$. It follows that the ideal $\mathbb{H}_{0,\mathfrak{h}} \cdot (\mathbb{C}[\mathfrak{h}]_+^{\Gamma_n} \otimes \mathbb{C}[\mathfrak{h}^*]_-^{\Gamma_n})$ of $\mathbb{H}_{0,\mathfrak{h}}$ must act trivially on M . Hence the action of $\mathbb{H}_{0,\mathfrak{h}}$ on M factors through the restricted rational Cherednik algebra $\overline{\mathbb{H}}_{0,\mathfrak{h}}$. In particular, M is a simple module over $\overline{\mathbb{H}}_{0,\mathfrak{h}}$.

Conversely, if N is a simple module over $\overline{\mathbb{H}}_{0,\mathfrak{h}}$, we can extend it to a simple module over $\mathbb{H}_{0,\mathfrak{h}}$ by means of the projection $\mathbb{H}_{0,\mathfrak{h}} \twoheadrightarrow \overline{\mathbb{H}}_{0,\mathfrak{h}}$. It is obvious that $\ker \chi_N \in \Upsilon^{-1}(0)$. \square

Therefore the task of describing the simple modules corresponding to the closed points of $(\text{Spec } Z_{0,\mathfrak{h}})^{\mathbb{C}^*}$ reduces to describing the simple modules over $\overline{\mathbb{H}}_{0,\mathfrak{h}}$.

4.2 Baby Verma modules. Let $\mathbb{C}[\mathfrak{h}]^{\text{co}\Gamma_n} := \mathbb{C}[\mathfrak{h}] / \mathbb{C}[\mathfrak{h}] \cdot \mathbb{C}[\mathfrak{h}]_+^{\Gamma_n}$ be the algebra of coinvariants with respect to the Γ_n -action. It follows from the PBW theorem for rational Cherednik algebras ([7, Theorem 1.3]) that there is an isomorphism of graded vector spaces $\overline{\mathbb{H}}_{0,\mathfrak{h}} \cong \mathbb{C}[\mathfrak{h}]^{\text{co}\Gamma_n} \otimes \mathbb{C}[\mathfrak{h}^*]^{\text{co}\Gamma_n} \otimes \mathbb{C}\Gamma_n$. Moreover, $\mathbb{C}[\mathfrak{h}^*]^{\text{co}\Gamma_n} \rtimes \mathbb{C}\Gamma_n$ is a subalgebra of $\overline{\mathbb{H}}_{0,\mathfrak{h}}$.

Definition 4.4. Let $\lambda \in \mathcal{P}(l, n)$. The irreducible $\mathbb{C}\Gamma_n$ -module $C(\lambda)$ becomes a module over $\mathbb{C}[\mathfrak{h}^*]^{co\Gamma_n} \rtimes \mathbb{C}\Gamma_n$ by means of the projection $\mathbb{C}[\mathfrak{h}^*]^{co\Gamma_n} \rtimes \mathbb{C}\Gamma_n \rightarrow \mathbb{C}\Gamma_n$. The *baby Verma module* associated to λ is the induced module

$$\Delta(\lambda) := \overline{\mathbb{H}}_{0, \mathfrak{h}} \otimes_{\mathbb{C}[\mathfrak{h}^*]^{co\Gamma_n} \rtimes \mathbb{C}\Gamma_n} C(\lambda).$$

It possesses a natural structure of a graded $\overline{\mathbb{H}}_{0, \mathfrak{h}}$ -module with $1 \otimes C(\lambda)$ in degree 0.

Theorem 4.5. Let $\lambda \in \mathcal{P}(l, n)$. The baby Verma module $\Delta(\lambda)$ is indecomposable with simple head $L(\lambda)$. Moreover, $\{L(\lambda) \mid \lambda \in \mathcal{P}(l, n)\}$ form a complete and irredundant set of representatives of isomorphism classes of graded simple $\overline{\mathbb{H}}_{0, \mathfrak{h}}$ -modules, up to a grading shift.

Proof. See [9, Proposition 4.3]. □

Corollary 4.6. There is a bijection

$$\mathcal{P}(l, n) \longleftrightarrow (\text{MaxSpec } Z_{0, \mathfrak{h}})^{\mathbb{C}^*}, \quad \lambda \mapsto \ker \chi_{L(\lambda)}.$$

Proof. This follows immediately from Theorem 4.5, Lemma 4.3 and Lemma 4.1. □

Notation. To simplify notation we will write χ_λ for $\chi_{L(\lambda)}$ and $\text{Ann}(\lambda)$ for $\ker \chi_{L(\lambda)}$. The latter notation is inspired by the fact that $\ker \chi_{L(\lambda)}$ is the ideal of annihilators of the simple module $L(\lambda)$ in $\mathbb{H}_{0, \mathfrak{h}}$.

5 The tautological vector bundle on $\text{Spec } Z_{0, \mathfrak{h}}$

In this section we consider the tautological vector bundle on $\text{Spec } Z_{0, \mathfrak{h}}$. We characterize it in two equivalent ways: as the coherent sheaf corresponding to the $e_n \mathbb{H}_{0, \mathfrak{h}} e_n$ -module $\mathbb{H}_{0, \mathfrak{h}} e_n$ and as the vector bundle induced by the trivial vector bundle on $\text{Rep}_{\mathbb{C}\Gamma_n}^o(\mathbb{H}_{0, \mathfrak{h}})$. We then consider the subbundle consisting of Γ_{n-1} -invariants.

5.1 The tautological vector bundle on $\text{Spec } Z_{0, \mathfrak{h}}$. Consider the $(\mathbb{H}_{0, \mathfrak{h}}, e_n \mathbb{H}_{0, \mathfrak{h}} e_n)$ -bimodule $\mathbb{H}_{0, \mathfrak{h}} e_n$. It is endowed with a \mathbb{C}^* -action induced by the \mathbb{C}^* -action on $\mathbb{H}_{0, \mathfrak{h}}$. The bimodule $\mathbb{H}_{0, \mathfrak{h}} e_n$ defines a \mathbb{C}^* -equivariant coherent sheaf $\widetilde{\mathbb{H}_{0, \mathfrak{h}} e_n}$ on $\text{Spec } e_n \mathbb{H}_{0, \mathfrak{h}} e_n \cong \text{Spec } Z_{0, \mathfrak{h}}$. The geometric fibre of this sheaf at the point $\ker \chi$ is $\mathbb{H}_{0, \mathfrak{h}} e_n \otimes_{e_n \mathbb{H}_{0, \mathfrak{h}} e_n} \mathbb{C}_\chi$, where $e_n \mathbb{H}_{0, \mathfrak{h}} e_n$ acts on \mathbb{C}_χ by the character χ . Note that each fibre is naturally a $\mathbb{H}_{0, \mathfrak{h}}$ -module as well as a \mathbb{C}^* -module (these actions are induced by the corresponding actions on $\mathbb{H}_{0, \mathfrak{h}}$). Since we are assuming that $\text{Spec } Z_{0, \mathfrak{h}}$ is smooth, Theorem 1.7 of [7] implies that the sheaf $\widetilde{\mathbb{H}_{0, \mathfrak{h}} e_n}$ is locally free.

Definition 5.1. Let \mathcal{R} denote the \mathbb{C}^* -equivariant vector bundle whose sheaf of sections is $\widetilde{\mathbb{H}_{0, \mathfrak{h}} e_n}$. We call it the *tautological vector bundle* on $\text{Spec } Z_{0, \mathfrak{h}}$.

5.2 Another description of \mathcal{R} . We recall another description of the vector bundle \mathcal{R} from [7, Proposition 3.8].

Definition 5.2. Let $\widehat{\mathcal{R}}$ denote the trivial vector bundle $\mathbb{C}\Gamma_n \times \text{Rep}_{\mathbb{C}\Gamma_n}^o(\mathbb{H}_{0, \mathfrak{h}}) \rightarrow \text{Rep}_{\mathbb{C}\Gamma_n}^o(\mathbb{H}_{0, \mathfrak{h}})$.

We let $\text{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n)$ act diagonally on the total space $\mathbb{C}\Gamma_n \times \text{Rep}_{\mathbb{C}\Gamma_n}^o(\mathbb{H}_{0, \mathfrak{h}})$, which makes $\widehat{\mathcal{R}}$ into a $\text{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n)$ -equivariant vector bundle. If $\phi \in \text{Rep}_{\mathbb{C}\Gamma_n}^o(\mathbb{H}_{0, \mathfrak{h}})$ let $\widehat{\mathcal{R}}_\phi$ denote the fibre of $\widehat{\mathcal{R}}$ at ϕ . Each fibre $\widehat{\mathcal{R}}_\phi$ possesses a canonical structure of an $\mathbb{H}_{0, \mathfrak{h}}$ -module, given by $z.v = \phi(z)(v)$ for all $z \in \mathbb{H}_{0, \mathfrak{h}}$ and $v \in \widehat{\mathcal{R}}_\phi \cong \mathbb{C}\Gamma_n$. The $\mathbb{H}_{0, \mathfrak{h}}$ -module structure commutes with the $\text{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n)$ -action.

Recall that we have defined a \mathbb{C}^* -action on $\text{Rep}_{\mathbb{C}\Gamma_n}^o(\mathbb{H}_{0, \mathfrak{h}})$. We make $\widehat{\mathcal{R}}$ into a \mathbb{C}^* -equivariant vector bundle by letting \mathbb{C}^* act trivially on the fibre $\mathbb{C}\Gamma_n$. The \mathbb{C}^* -action also commutes with the $\text{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n)$ -action.

Let

$$\text{PAut}_{\Gamma_n}(\mathbb{C}\Gamma_n) := \text{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n) / \mathbb{C}^*$$

be the quotient modulo the scalars. We want to endow $\widehat{\mathcal{R}}$ with the structure of a $\mathrm{PAut}_{\Gamma_n}(\mathbb{C}\Gamma_n)$ -equivariant vector bundle. The $\mathrm{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n)$ -action on $\mathrm{Rep}_{\mathbb{C}\Gamma_n}^o(\mathbb{H}_{0,\mathbf{h}})$ factors through the canonical projection $\mathrm{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n) \rightarrow \mathrm{PAut}_{\Gamma_n}(\mathbb{C}\Gamma_n)$. However, the $\mathrm{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n)$ -action on $\mathbb{C}\Gamma_n$ doesn't factor through this projection. To circumvent this problem, we define a splitting

$$\mathrm{PAut}_{\Gamma_n}(\mathbb{C}\Gamma_n) \hookrightarrow \mathrm{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n) \quad (9)$$

of the projection by the rule that the image of $\mathrm{PAut}_{\Gamma_n}(\mathbb{C}\Gamma_n)$ under (9) acts trivially on $e_n \in \mathbb{C}\Gamma_n$. The splitting defines a $\mathrm{PAut}_{\Gamma_n}(\mathbb{C}\Gamma_n)$ -action on $\mathbb{C}\Gamma_n$, which makes $\widehat{\mathcal{R}}$ into a $\mathrm{PAut}_{\Gamma_n}(\mathbb{C}\Gamma_n)$ -equivariant vector bundle.

Lemma 5.3. *The trivial vector bundle $\widehat{\mathcal{R}}$ descends to a \mathbb{C}^* -equivariant vector bundle*

$$\mathbb{C}\Gamma_n \times^{\mathrm{PAut}_{\Gamma_n}(\mathbb{C}\Gamma_n)} \mathrm{Rep}_{\mathbb{C}\Gamma_n}^o(\mathbb{H}_{0,\mathbf{h}}) \rightarrow \mathrm{Rep}_{\mathbb{C}\Gamma_n}^o(\mathbb{H}_{0,\mathbf{h}}) // \mathrm{PAut}_{\Gamma_n}(\mathbb{C}\Gamma_n) \cong \mathrm{Spec} Z_{0,\mathbf{h}}. \quad (10)$$

Proof. Since we are assuming that $\mathrm{Spec} Z_{0,\mathbf{h}}$ is smooth, the lemma follows from the proof of Proposition 3.8 in [7]. \square

Definition 5.4. We denote the vector bundle (10) by $\widetilde{\mathcal{R}}$. Let \widetilde{e}_n be the non-vanishing regular section of $\widetilde{\mathcal{R}}$ induced by the constant section of $\widehat{\mathcal{R}}$ with value e_n .

Proposition 5.5. *The map of $\mathbb{H}_{0,\mathbf{h}}$ -modules $\Psi : \mathbb{H}_{0,\mathbf{h}} e_n \rightarrow \Gamma(\mathrm{Spec} Z_{0,\mathbf{h}}, \widetilde{\mathcal{R}})$, $z \cdot e_n \mapsto z \cdot \widetilde{e}_n$ induces a $(\mathbb{H}_{0,\mathbf{h}}, \mathbb{C}^*)$ -equivariant isomorphism of vector bundles $\mathcal{R} \xrightarrow{\cong} \widetilde{\mathcal{R}}$.*

Proof. The fact that the induced map $\mathcal{R} \rightarrow \widetilde{\mathcal{R}}$ is an $\mathbb{H}_{0,\mathbf{h}}$ -equivariant isomorphism is shown in the proof of Proposition 3.8 in [7]. We prove \mathbb{C}^* -equivariance. It suffices to show that the map Ψ is \mathbb{C}^* -equivariant. Thus if $t \in \mathbb{C}^*$ and $z \in \mathbb{H}_{0,\mathbf{h}}$, we need to show that $(t.z) \cdot \widetilde{e}_n = t.(z \cdot \widetilde{e}_n)$. Let $\alpha := (t.z) \cdot \widetilde{e}_n$ and $\beta := t.(z \cdot \widetilde{e}_n)$. A group element $t \in \mathbb{C}^*$ acts on $\Gamma(\mathrm{Spec} Z_{0,\mathbf{h}}, \widetilde{\mathcal{R}})$ by sending a section s to the section

$$s' : \mathrm{Spec} Z_{0,\mathbf{h}} \rightarrow \mathbb{C}\Gamma_n \times^{\mathrm{PAut}_{\Gamma_n}(\mathbb{C}\Gamma_n)} \mathrm{Rep}_{\mathbb{C}\Gamma_n}^o(\mathbb{H}_{0,\mathbf{h}}), \quad \phi \mapsto t.(s(t^{-1}.\phi)).$$

On the other hand, an element $z \in \mathbb{H}_{0,\mathbf{h}}$ acts on $\Gamma(\mathrm{Spec} Z_{0,\mathbf{h}}, \widetilde{\mathcal{R}})$ by sending a section s to the section

$$s' : \mathrm{Spec} Z_{0,\mathbf{h}} \rightarrow \mathbb{C}\Gamma_n \times^{\mathrm{PAut}_{\Gamma_n}(\mathbb{C}\Gamma_n)} \mathrm{Rep}_{\mathbb{C}\Gamma_n}^o(\mathbb{H}_{0,\mathbf{h}}), \quad \phi \mapsto z.s(\phi).$$

Therefore $\beta(\phi) = t.[z.e_n, t^{-1}.\widetilde{\phi}]$ for some $\widetilde{\phi} \in \mathrm{Rep}_{\mathbb{C}\Gamma_n}^o(\mathbb{H}_{0,\mathbf{h}})$ lifting ϕ (i.e. $\pi(\widetilde{\phi}) = \phi$), where $[z.e_n, t^{-1}.\widetilde{\phi}]$ is the $\mathrm{PAut}_{\Gamma_n}(\mathbb{C}\Gamma_n)$ -orbit of the point $(z.e_n, t^{-1}.\widetilde{\phi})$. We regard $\widetilde{\phi}$ as an algebra homomorphism $\widetilde{\phi} : \mathbb{H}_{0,\mathbf{h}} \rightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}\Gamma_n)$. Let us set $\widetilde{\phi}_z := \widetilde{\phi}(z) \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}\Gamma_n)$. We have

$$\beta(\phi) = t.[z.e_n, t^{-1}.\widetilde{\phi}] = t.[(t^{-1}.\widetilde{\phi})_z(e_n), t^{-1}.\widetilde{\phi}] = t.[\widetilde{\phi}_{t.z}(e_n), t^{-1}.\widetilde{\phi}] = [\widetilde{\phi}_{t.z}(e_n), \widetilde{\phi}]$$

while $\alpha(\phi) = [(t.z).e_n, \widetilde{\phi}] = [\widetilde{\phi}_{t.z}(e_n), \widetilde{\phi}]$. This completes the proof of \mathbb{C}^* -equivariance. \square

5.3 The vector bundle $\mathcal{R}^{\Gamma_{n-1}}$. We are especially interested in the subbundle of \mathcal{R} consisting of Γ_{n-1} -invariants. We are now going to characterize in several equivalent ways.

Definition 5.6. The group $\mathbb{C}\Gamma_n$ acts naturally on every fibre of \mathcal{R} from the left. Recall the idempotent $e_{n-1} = \frac{1}{l^{n-1}(n-1)!} \sum_{g \in \Gamma_{n-1}} g \in \mathbb{C}\Gamma_n$. We set

$$\mathcal{R}^{\Gamma_{n-1}} = e_{n-1} \mathcal{R}.$$

We let $\mathcal{R}(\chi) := \mathrm{Hom}_{\Gamma_n}(\chi, \mathcal{R})$ denote the χ -isotypic component of the vector bundle \mathcal{R} . Let $\mathrm{ind} := \mathrm{Ind}_{\Gamma_{n-1}}^{\Gamma_n} \mathrm{triv}$. By Frobenius reciprocity we have

$$\begin{aligned} \mathcal{R}(\mathrm{ind}) &= \mathrm{Hom}_{\Gamma_n}(\mathrm{ind}, \mathcal{R}) = \mathrm{Hom}_{\Gamma_n}(\mathrm{Ind}_{\Gamma_{n-1}}^{\Gamma_n} \mathrm{triv}, \mathcal{R}) = \mathrm{Hom}_{\Gamma_{n-1}}(\mathrm{triv}, \mathcal{R}|_{\Gamma_{n-1}}) \\ &= \mathrm{triv} \otimes_{\Gamma_{n-1}} \mathcal{R} = e_{n-1} \mathcal{R} = \mathcal{R}^{\Gamma_{n-1}}. \end{aligned}$$

We can also characterize $\mathcal{R}^{\Gamma_{n-1}}$ as the vector bundle whose sheaf of sections is the coherent sheaf associated to the right $e_n \mathbb{H}_{0,\mathbf{h}} e_n$ -module $e_{n-1} \mathbb{H}_{0,\mathbf{h}} e_n$. Finally, we observe that the action of the group $\mathrm{PAut}_{\Gamma_n}(\mathbb{C}\Gamma_n)$ on $\mathbb{C}\Gamma_n$ restricts to an action on the subspace $e_{n-1} \mathbb{C}\Gamma_n$. Hence the vector bundle $\widetilde{\mathcal{R}}^{\Gamma_{n-1}}$:

$$e_{n-1} \mathbb{C}\Gamma_n \times^{\mathrm{PAut}_{\Gamma_n}(\mathbb{C}\Gamma_n)} \mathrm{Rep}_{\mathbb{C}\Gamma_n}(\mathbb{H}_{0,\mathbf{h}}) \rightarrow \mathrm{Rep}_{\mathbb{C}\Gamma_n}(\mathbb{H}_{0,\mathbf{h}}) / \mathrm{PAut}_{\Gamma_n}(\mathbb{C}\Gamma_n) \cong \mathrm{Spec} Z_{0,\mathbf{h}}$$

is well-defined. An obvious modification of the proof of Proposition 5.5 shows that there is a \mathbb{C}^* -equivariant isomorphism of vector bundles $\mathcal{R}^{\Gamma_{n-1}} \xrightarrow{\cong} \widetilde{\mathcal{R}}^{\Gamma_{n-1}}$.

Definition 5.7. For each $\underline{\lambda} \in \mathcal{P}(l, n)$ let $(\mathcal{R}^{\Gamma_{n-1}})_{\underline{\lambda}}$ denote the fibre of the vector bundle $\mathcal{R}^{\Gamma_{n-1}}$ at the fixed point $\mathrm{Ann}(\underline{\lambda})$.

We have

$$(\mathcal{R}^{\Gamma_{n-1}})_{\underline{\lambda}} = e_{n-1} \mathbb{H}_{0,\mathbf{h}} e_n \otimes_{e_n \mathbb{H}_{0,\mathbf{h}} e_n} \mathbb{C}_{\chi_{\underline{\lambda}}}.$$

6 Combinatorics I

Our next goal is to calculate the \mathbb{C}^* -characters of the fibres $(\mathcal{R}^{\Gamma_{n-1}})_{\underline{\lambda}}$. Before doing so, we need to recall some combinatorics. In this section we introduce the notation for Young tableaux, contents and residues. We also recall the definition of hook length polynomials.

6.1 Partition statistics and q -analogs. We define the following "partition statistics". If $\lambda = (\lambda_1, \lambda_2, \dots)$ is a partition of k we set $n(\lambda) := \sum_{i \geq 1} i \cdot \lambda_{i+1}$. If $\underline{\lambda} = (\lambda^0, \dots, \lambda^{l-1})$ is an l -multipartition of k we define $r(\underline{\lambda}) = \sum_{i=1}^{l-1} i \cdot |\lambda^i|$. We also recall the notations $[n]_t = \frac{1-t^n}{1-t} = 1 + \dots + t^{n-1}$, $(t)_n = (1-t)(1-t^2)\dots(1-t^n)$.

6.2 Young tableaux. Let $\lambda = (\lambda_1, \dots, \lambda_m, 0, \dots)$ be a partition of k , where $\lambda_1, \dots, \lambda_m$ are non-zero. Let $\mathbb{Y}(\mu) := \{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq \lambda_i\}$ denote the Young diagram of μ . We will always display Young diagrams according to the English convention. We call each pair $(i, j) \in \mathbb{Y}(\mu)$ a cell. We will often use the symbol \square to refer to cells. Sometimes we will also abuse notation and write μ instead of $\mathbb{Y}(\mu)$ where no confusion can arise, e.g., $\square \in \mu$ instead of $\square \in \mathbb{Y}(\mu)$.

Now suppose that $\underline{\lambda} = (\lambda^0, \dots, \lambda^{l-1})$ is an l -multipartition of k . By the Young diagram of $\underline{\lambda}$ we mean the l -tuple $(\mathbb{Y}(\lambda^0), \dots, \mathbb{Y}(\lambda^{l-1}))$.

6.3 Contents and residues. If $\square = (i, j) \in \mathbb{Y}(\lambda)$ is a cell, let $c(\square) := j - i$ be the *content* of \square . We call $\mathrm{Res}_{\lambda}(t) := \sum_{\square \in \lambda} t^{c(\square)}$ the *residue* of λ . We also call $c(\square) \bmod l$ the l -*content* of \square and $\sum_{\square \in \lambda} t^{c(\square) \bmod l}$ the l -*residue* of λ . It is clear that a partition is determined uniquely by its residue.

Now suppose that $\underline{\lambda}$ is an l -multipartition. Let $\mathbf{s} = (s_0, \dots, s_{l-1}) \in \mathbb{Q}^l$. We define the \mathbf{s} -residue of $\underline{\lambda}$ to be

$$\mathrm{Res}_{\underline{\lambda}}^{\mathbf{s}}(t) := \sum_{i=0}^{l-1} t^{s_i} \mathrm{Res}_{\lambda^i}(t).$$

For sufficiently generic \mathbf{s} , an l -multipartition is determined uniquely by its \mathbf{s} -residue.

6.4 Hooks and hook length polynomials. Let μ be a partition and fix a cell $\square = (i, j) \in \mathbb{Y}(\mu)$. By the *hook* associated to the cell (i, j) we mean the set $\{(i, j)\} \cup \{(i', j) \in \mathbb{Y}(\mu) \mid i' > i\} \cup \{(i, j') \in \mathbb{Y}(\mu) \mid j' > j\}$. We call (i, j) the *root* of the hook, $\{(i', j) \in \mathbb{Y}(\mu) \mid i' > i\}$ the *leg* of the hook and $\{(i, j') \in \mathbb{Y}(\mu) \mid j' > j\}$ the *arm* of the hook. The cell in the leg of the hook with the largest first coordinate is called the *foot* of the hook, and the cell in the arm of the hook with the largest second coordinate is called the *hand* of the hook.

By a *hook* in $\mathbb{Y}(\mu)$ we mean a hook associated to some cell $(i, j) \in \mathbb{Y}(\mu)$. If H is a hook, let $\text{arm}(H)$ denote its arm and let $\text{leg}(H)$ denote its leg.

Consider again the cell $\square = (i, j)$. Let $a_\lambda(\square)$ denote the number of cells in the arm of the hook associated to \square and let $l_\lambda(\square)$ denote the number of cells in the leg of the hook associated to \square . The *hook length* of \square is defined to be $h_\lambda(\square) := 1 + a_\lambda(\square) + l_\lambda(\square)$, which equals the total number of cells in the hook associated to \square .

The *hook length polynomial* of the partition λ is

$$H_\lambda(t) = \prod_{\square \in \lambda} (1 - t^{h_\lambda(\square)}).$$

Hook length polynomials are related to Schur functions by the following equality

$$s_\lambda(1, t, t^2, \dots) = \frac{t^{n(\lambda)}}{H_\lambda(t)}.$$

6.5 Frobenius hooks. By a *Frobenius hook* in $\mathbb{Y}(\mu)$ we mean a hook whose root is a cell of content zero. Clearly $\mathbb{Y}(\mu)$ is the disjoint union of all its Frobenius hooks. Suppose that $(1, 1), (2, 2), \dots, (k, k)$ are the cells of content zero in $\mathbb{Y}(\mu)$. Let F_i denote the Frobenius hook with root (i, i) . We endow the set of Frobenius hooks with the natural ordering $F_1 < F_2 < \dots < F_k$. We call F_1 the *innermost* or *first* Frobenius hook and F_k the *outermost* or *last* Frobenius hook.

7 Calculation of characters

In this section we will calculate the \mathbb{C}^* -characters of the fibres of the vector bundle $\mathcal{R}^{\Gamma_{n-1}}$ at the \mathbb{C}^* -fixed points.

7.1 The strategy. We will first identify the graded vector space $(\mathcal{R}^{\Gamma_{n-1}})_\lambda$ with a graded shift of $e_{n-1}L(\lambda)$. This reduces our task to the calculation of the \mathbb{C}^* -character of $e_{n-1}L(\lambda)$. We will split this problem into two parts. We first calculate the graded multiplicity with which $L(\lambda)$ occurs in $\Delta(\lambda)$. We then calculate the character of $e_{n-1}\Delta(\lambda)$ and use the equation

$$\text{ch}_t e_{n-1}L(\lambda) = \frac{\text{ch}_t e_{n-1}\Delta(\lambda)}{[\Delta(\lambda) : L(\lambda)]}.$$

The calculation of $\text{ch}_t e_{n-1}\Delta(\lambda)$ is rather involved. We show that there exists an isomorphism of graded vector spaces

$$e_{n-1}\Delta(\lambda) \cong \left(\bigoplus_{\mu \uparrow \lambda} e_{n-1}\Delta(\mu) \right) \otimes U,$$

where U is a graded vector space with character equal to $[ln]_t$. We finally prove that both $\text{ch}_t e_{n-1}\Delta(\mu)$ and $[\Delta(\lambda) : L(\lambda)]$ are given by fake degree polynomials.

7.2 $(\mathcal{R}^{\Gamma_{n-1}})_\lambda$ as a shift of $e_{n-1}L(\lambda)$. Let $q(\lambda)$ be the degree in which the trivial Γ_n -module triv occurs in $L(\lambda)$.

Lemma 7.1. *We have a graded $\mathbb{H}_{0, \mathbf{h}}$ -module isomorphism*

$$\mathcal{R}_\lambda = \mathbb{H}_{0, \mathbf{h}} e_n \otimes_{e_n \mathbb{H}_{0, \mathbf{h}} e_n} \mathbb{C}_{\chi_\lambda} \cong L(\lambda)[t^{-q(\lambda)}].$$

Proof. By definition, χ_λ is the character by which $\text{Spec } Z_{0, \mathbf{h}} \cong e_n \mathbb{H}_{0, \mathbf{h}} e_n$ acts on $L(\lambda)$. Hence $\mathfrak{m} := \ker \chi_\lambda \subset Z_{0, \mathbf{h}}$ annihilates $L(\lambda)$. On the other hand, we also have

$$\mathfrak{m} \cdot \mathbb{H}_{0, \mathbf{h}} e_n \otimes_{e_n \mathbb{H}_{0, \mathbf{h}} e_n} \mathbb{C}_{\chi_\lambda} = \mathbb{H}_{0, \mathbf{h}} e_n \otimes_{e_n \mathbb{H}_{0, \mathbf{h}} e_n} e_n \mathfrak{m} \mathbb{C}_{\chi_\lambda} = \{0\}$$

because $\mathbb{C}_{\chi_{\underline{\lambda}}} \cong e_n \mathbb{H}_{0, \mathbf{h}} e_n / e_n \mathbf{m}$. Since \mathbf{m} annihilates both the simple $\mathbb{H}_{0, \mathbf{h}}$ -modules $L(\underline{\lambda})$ and $\mathbb{H}_{0, \mathbf{h}} e_n \otimes_{e_n \mathbb{H}_{0, \mathbf{h}} e_n} \mathbb{C}_{\chi_{L(\underline{\lambda})}}$, it follows that they must be isomorphic up to a shift in the grading. To determine this shift, we consider the degree of the trivial Γ_n -representation triv in both modules. By the definition of the number $q(\underline{\lambda})$, triv occurs in $L(\underline{\lambda})$ in degree $q(\underline{\lambda})$. On the other hand, we can identify triv with the subspace $e_n \otimes_{e_n \mathbb{H}_{0, \mathbf{h}} e_n} \mathbb{C}_{\chi_{\underline{\lambda}}}$ so triv occurs in degree zero in $\mathbb{H}_{0, \mathbf{h}} e_n \otimes_{e_n \mathbb{H}_{0, \mathbf{h}} e_n} \mathbb{C}_{\chi_{\underline{\lambda}}}$. Therefore $\mathbb{H}_{0, \mathbf{h}} e_n \otimes_{e_n \mathbb{H}_{0, \mathbf{h}} e_n} \mathbb{C}_{\chi_{\underline{\lambda}}} \cong L(\underline{\lambda})[t^{-q(\underline{\lambda})}]$. \square

It follows that there is a graded vector space isomorphism

$$(\mathcal{R}^{\Gamma_{n-1}})_{\underline{\lambda}} = e_{n-1} \mathcal{R}_{\underline{\lambda}} \cong e_n L(\underline{\lambda})[t^{-q(\underline{\lambda})}].$$

7.3 Coinvariants algebras and fake degree polynomials. In our calculations we will repeatedly need to determine the graded multiplicity with which a certain Γ_n - or Γ_{n-1} -module occurs in the coinvariants algebra $\mathbb{C}[\mathfrak{h}]^{co\Gamma_n}$. These multiplicities are given by fake degree polynomials.

Definition 7.2. Let $\mathbb{C}[\mathfrak{h}]^{co\Gamma_n} := \mathbb{C}[\mathfrak{h}] / \mathbb{C}[\mathfrak{h}] \cdot \mathbb{C}[\mathfrak{h}]_+^{\Gamma_n}$ be the *algebra of coinvariants*.

The algebra $\mathbb{C}[\mathfrak{h}]^{co\Gamma_n}$ carries a natural structure of a graded Γ_n -module. It is well known that $\mathbb{C}[\mathfrak{h}]^{co\Gamma_n}$ is isomorphic to the regular representation $\mathbb{C}\Gamma_n$ as an ungraded Γ_n -module.

Definition 7.3. Let $\mathbb{C}[\mathfrak{h}]^{co\Gamma_n}|_{\Gamma_{n-1}}$ denote the space of coinvariants $\mathbb{C}[\mathfrak{h}]^{co\Gamma_n}$ considered as a Γ_{n-1} -module by means of the inclusion $\Gamma_{n-1} \hookrightarrow \Gamma_n$. Let $\mathfrak{h}' \subset \mathfrak{h}$ denote the subspace spanned by y_2, \dots, y_n .

Lemma 7.4. *We have an isomorphism of graded Γ_{n-1} -modules*

$$\mathbb{C}[\mathfrak{h}]^{co\Gamma_n}|_{\Gamma_{n-1}} \cong \mathbb{C}[\mathfrak{h}']^{co\Gamma_{n-1}} \otimes U,$$

where U is a graded vector space with Poincaré polynomial $\text{ch}_t U = [nl]_t$.

Proof. We have a sequence of inclusions of graded Γ_{n-1} -modules

$$\mathbb{C}[\mathfrak{h}]^{\Gamma_n} \hookrightarrow \mathbb{C}[\mathfrak{h}]^{\Gamma_{n-1}} \hookrightarrow \mathbb{C}[\mathfrak{h}]$$

such that each ring is a free graded module over the previous ring. Hence there is an isomorphism of graded Γ_{n-1} -modules

$$\mathbb{C}[\mathfrak{h}] / \langle \mathbb{C}[\mathfrak{h}]^{\Gamma_n} \rangle \cong \mathbb{C}[\mathfrak{h}] / \langle \mathbb{C}[\mathfrak{h}]^{\Gamma_{n-1}} \rangle \otimes \mathbb{C}[\mathfrak{h}]^{\Gamma_{n-1}} / \langle \mathbb{C}[\mathfrak{h}]^{\Gamma_n} \rangle.$$

Observe that there is also an isomorphism of graded Γ_{n-1} -modules

$$\mathbb{C}[\mathfrak{h}] / \langle \mathbb{C}[\mathfrak{h}]^{\Gamma_{n-1}} \rangle \cong \mathbb{C}[\mathfrak{h}'] / \langle \mathbb{C}[\mathfrak{h}']^{\Gamma_{n-1}} \rangle = \mathbb{C}[\mathfrak{h}']^{co\Gamma_{n-1}}.$$

To prove the lemma it now suffices to find the Poincaré polynomial of the graded vector space $\mathbb{C}[\mathfrak{h}]^{\Gamma_{n-1}} / \langle \mathbb{C}[\mathfrak{h}]^{\Gamma_n} \rangle$. We know that $\mathbb{C}[\mathfrak{h}]^{\Gamma_{n-1}}$ is a polynomial algebra with generators in degrees $l, 2l, \dots, (n-1)l$ and an additional generator in degree 1. The ring $\mathbb{C}[\mathfrak{h}]^{\Gamma_n}$ is a polynomial algebra with generators in degrees $l, 2l, \dots, nl$. Hence

$$\text{ch}_t \mathbb{C}[\mathfrak{h}]^{\Gamma_{n-1}} = \frac{1}{1-t} \prod_{i=1}^{n-1} \frac{1}{1-t^{il}}, \quad \mathbb{C}[\mathfrak{h}]^{\Gamma_n} = \prod_{i=1}^n \frac{1}{1-t^{il}}.$$

It follows that $\text{ch}_t \mathbb{C}[\mathfrak{h}]^{\Gamma_{n-1}} / \langle \mathbb{C}[\mathfrak{h}]^{\Gamma_n} \rangle = \frac{\text{ch}_t \mathbb{C}[\mathfrak{h}]^{\Gamma_{n-1}}}{\text{ch}_t \mathbb{C}[\mathfrak{h}]^{\Gamma_n}} = \frac{1-t^{nl}}{1-t} = [nl]_t$. \square

Definition 7.5. Suppose that we are given an l -multipartition $\underline{\lambda} \in \mathcal{P}(l, n)$ and the corresponding irreducible representation $C(\underline{\lambda})$ of Γ_n . We regard $C(\underline{\lambda})$ as a graded Γ_n -module concentrated in degree zero. We define the *fake degree polynomial* associated to $\underline{\lambda}$ to be

$$f_{\underline{\lambda}}(t) := \sum_{k \in \mathbb{Z}} [\mathbb{C}[\mathfrak{h}]^{co\Gamma_n} : C(\underline{\lambda})^*[k]] t^k.$$

Theorem 7.6. *Let $\underline{\lambda} \in \mathcal{P}(l, n)$. We have*

$$f_{\underline{\lambda}}(t) = t^{r(\underline{\lambda})}(t^l)_n \prod_{i=0}^{l-1} \frac{t^{l \cdot n(\lambda^i)}}{H_{\lambda^i}(t^l)} = t^{r(\underline{\lambda})}(t^l)_n \prod_{i=0}^{l-1} s_{\lambda^i}(1, t^l, t^{2l}, \dots).$$

In particular, if λ is a partition of n then $f_{\lambda} = (t)_n \frac{t^{n(\lambda)}}{H_{\lambda}(t)} = (t)_n s_{\lambda}(1, t, t^2, \dots)$.

Proof. See [27, Theorem 5.3]. □

7.4 The graded multiplicity of $L(\underline{\lambda})$ in $\Delta(\underline{\lambda})$. We will now express the graded multiplicity with which $L(\underline{\lambda})$ occurs in $\Delta(\underline{\lambda})$ as a shift of a fake degree polynomial.

Lemma 7.7. *The algebra $\overline{\mathbb{H}}_{0, \mathbf{h}}$ has a block decomposition*

$$\overline{\mathbb{H}}_{0, \mathbf{h}} = \bigoplus_{\underline{\lambda} \in \mathcal{P}(l, n)} \mathcal{B}_{\underline{\lambda}},$$

where each $\mathcal{B}_{\underline{\lambda}}$ is an indecomposable algebra with a unique simple module $L(\underline{\lambda})$.

Proof. This follows from [9, §5.3] and the fact that $\text{Spec } Z_{0, \mathbf{h}}$ is smooth. □

Lemma 7.8. *Let $\underline{\lambda} \in \mathcal{P}(l, n)$. The simple $\overline{\mathbb{H}}_{0, \mathbf{h}}$ -module $L(\underline{\lambda})$ occurs in $\Delta(\underline{\lambda})$ with graded multiplicity*

$$\sum_{k \in \mathbb{Z}} [\Delta(\underline{\lambda}) : L(\underline{\lambda})[k]] t^k = t^{-q(\underline{\lambda})} f_{\underline{\lambda}}(t),$$

where $q(\underline{\lambda})$ is the degree in which the trivial Γ_n -module triv occurs in $L(\underline{\lambda})$.

Proof. Lemma 7.7 and the fact that $\Delta(\underline{\lambda})$ is indecomposable imply that all the composition factors of $\Delta(\underline{\lambda})$ are shifts of the simple module $L(\underline{\lambda})$. It follows from the definition of $\Delta(\underline{\lambda})$ as an induced module that we have a graded vector space isomorphism $\Delta(\underline{\lambda}) \cong \mathbb{C}[\mathbf{h}]^{co\Gamma_n} \otimes C(\underline{\lambda})$, where $1 \otimes C(\underline{\lambda})$ has degree zero. Hence $\dim \Delta(\underline{\lambda}) = |\Gamma_n| \cdot \dim C(\underline{\lambda})$. Also recall that since $\text{Spec } Z_{0, \mathbf{h}}$ is smooth, $\dim L(\underline{\lambda}) = |\Gamma_n|$. Therefore in the graded Grothendieck group of $\overline{\mathbb{H}}_{0, \mathbf{h}}$ we have

$$[\Delta(\underline{\lambda})] = [L(\underline{\lambda})][i_1] + [L(\underline{\lambda})][i_2] + \dots + [L(\underline{\lambda})][i_m], \quad (11)$$

with $m = \dim C(\underline{\lambda})$ and $i_1, i_2, \dots, i_m \in \mathbb{Z}$. Recall that $L(\underline{\lambda})$ is isomorphic to the regular representation $\mathbb{C}\Gamma_n$ as a Γ_n -module. Hence the trivial representation triv of Γ_n occurs in $L(\underline{\lambda})$ with multiplicity one. Let $q(\underline{\lambda})$ denote the degree in which triv occurs in $L(\underline{\lambda})$. Then triv occurs on the RHS of (11) with graded multiplicity $t^{q(\underline{\lambda})}(t^{i_1} + t^{i_2} + \dots + t^{i_m})$. On the other hand, $\Delta(\underline{\lambda})$ is isomorphic to $\mathbb{C}[\mathbf{h}]^{co\Gamma_n} \otimes C(\underline{\lambda})$ as a graded Γ_n -module. Let $\mathbb{C}[\mathbf{h}]^{co\Gamma_n} \cong \bigoplus_i U_i$ be the decomposition into simple graded Γ_n -modules. Then $U_i \otimes C(\underline{\lambda}) \cong \text{Hom}_{\mathbb{C}}(U_i^*, C(\underline{\lambda}))$ contains the trivial representation if and only if $U_i = C(\underline{\lambda})^*[k]$ for some $k \in \mathbb{Z}$, in which case the multiplicity of the trivial representation is one. Hence the graded multiplicity of triv in $\Delta(\underline{\lambda})$ is

$$\sum_{k \in \mathbb{Z}} [\Delta(\underline{\lambda}) : \text{triv}[k]] t^k = \sum_{k \in \mathbb{Z}} [\mathbb{C}[\mathbf{h}]^{co\Gamma_n} : C(\underline{\lambda})^*[k]] t^k = f_{\underline{\lambda}}(t).$$

It follows that $(t^{i_1} + t^{i_2} + \dots + t^{i_m}) = t^{-q(\underline{\lambda})} f_{\underline{\lambda}}(t)$. □

7.5 The character of $\text{ch}_t e_{n-1} \Delta(\underline{\lambda})$. We will now calculate the character of $\text{ch}_t e_{n-1} \Delta(\underline{\lambda})$ and express it as a sum of fake degree polynomials.

Lemma 7.9. *We have*

$$\text{ch}_t e_{n-1} \Delta(\underline{\lambda}) = [ln]_t \sum_{\underline{\mu} \uparrow \underline{\lambda}} f_{\underline{\mu}}(t).$$

Proof. By Lemma 7.4 and Proposition 2.1, we have isomorphisms of graded Γ_{n-1} -modules

$$\begin{aligned}\Delta(\underline{\lambda})|_{\Gamma_{n-1}} &\cong \mathbb{C}[\mathfrak{h}]^{co\Gamma_n}|_{\Gamma_{n-1}} \otimes C(\underline{\lambda})|_{\Gamma_{n-1}} \\ &\cong (\mathbb{C}[\mathfrak{h}']^{co\Gamma_{n-1}} \otimes U) \otimes \left(\bigoplus_{\underline{\mu} \uparrow \underline{\lambda}} C(\underline{\mu}) \right) \cong \left(\bigoplus_{\underline{\mu} \uparrow \underline{\lambda}} \Delta(\underline{\mu}) \right) \otimes U,\end{aligned}$$

where U is a graded vector space with character $\text{ch}_t U = [ln]_t$. Hence

$$e_{n-1}\Delta(\underline{\lambda}) \cong \left(\bigoplus_{\underline{\mu} \uparrow \underline{\lambda}} e_{n-1}\Delta(\underline{\mu}) \right) \otimes U \quad (12)$$

as graded Γ_{n-1} -modules. We now compute the \mathbb{C}^* -character of each summand $e_{n-1}\Delta(\underline{\mu})$. First of all recall that there is a graded Γ_{n-1} -module isomorphism $e_{n-1}\Delta(\underline{\mu}) \cong e_{n-1}(\mathbb{C}[\mathfrak{h}']^{co\Gamma_{n-1}} \otimes C(\underline{\mu}))$. Suppose that $\mathbb{C}[\mathfrak{h}']^{co\Gamma_{n-1}} \otimes C(\underline{\mu}) \cong \bigoplus_i W_i$ is the decomposition of the tensor product on the LHS into graded simple Γ_{n-1} -modules. Each $e_{n-1}W_i$ is a subrepresentation of W_i on which Γ_{n-1} acts trivially. The simplicity of W_i now implies that either W_i is isomorphic to the trivial representation, in which case $e_{n-1}W_i = W_i$, or $e_{n-1}W_i = \{0\}$. Therefore $\text{ch}_t e_{n-1}\Delta(\underline{\mu}) = \sum_{k \in \mathbb{Z}} [\mathbb{C}[\mathfrak{h}']^{co\Gamma_{n-1}} \otimes C(\underline{\mu}) : \text{triv}[k]] t^k$. Now let $\mathbb{C}[\mathfrak{h}']^{co\Gamma_{n-1}} \cong \bigoplus_i U_i$ be the decomposition into simple graded Γ_{n-1} -modules. Then $U_i \otimes C(\underline{\mu}) \cong \text{Hom}_{\mathbb{C}}(U_i^*, C(\underline{\mu}))$ contains the trivial representation if and only if $U_i \cong C(\underline{\mu})^*[k]$ for some $k \in \mathbb{Z}$, in which case the multiplicity of the trivial representation is one. Hence

$$\text{ch}_t e_{n-1}\Delta(\underline{\mu}) = \sum_{k \in \mathbb{Z}} [\mathbb{C}[\mathfrak{h}']^{co\Gamma_{n-1}} : C(\underline{\mu})^*[k]] t^k = f_{\underline{\mu}}(t). \quad (13)$$

Combining (12) with (13) we obtain $\text{ch}_t e_{n-1}\Delta(\underline{\lambda}) = [ln]_t \sum_{\underline{\mu} \uparrow \underline{\lambda}} f_{\underline{\mu}}(t)$. \square

7.6 The character of $(\mathcal{R}^{\Gamma_{n-1}})_{\underline{\lambda}}$. We can now put together the calculations we have performed so far to obtain the character of $(\mathcal{R}^{\Gamma_{n-1}})_{\underline{\lambda}}$.

Theorem 7.10. *Let $\underline{\lambda} \in \mathcal{P}(l, n)$. Then*

$$\text{ch}_t(\mathcal{R}^{\Gamma_{n-1}})_{\underline{\lambda}} = [ln]_t \sum_{\underline{\mu} \uparrow \underline{\lambda}} \frac{f_{\underline{\mu}}(t)}{f_{\underline{\lambda}}(t)}.$$

Proof. By Lemma 7.1, Lemma 7.8 and Lemma 7.9, we have

$$\begin{aligned}\text{ch}_t(\mathcal{R}^{\Gamma_{n-1}})_{\underline{\lambda}} &= t^{-q(\underline{\lambda})} \cdot \text{ch}_t e_{n-1}L(\underline{\lambda}) \\ &= (t^{-q(\underline{\lambda})} \cdot \text{ch}_t e_{n-1}\Delta(\underline{\lambda})) / (t^{-q(\underline{\lambda})} f_{\underline{\lambda}}(t)) \\ &= \text{ch}_t e_{n-1}\Delta(\underline{\lambda}) / f_{\underline{\lambda}}(t) = [ln]_t \sum_{\underline{\mu} \uparrow \underline{\lambda}} \frac{f_{\underline{\mu}}(t)}{f_{\underline{\lambda}}(t)}.\end{aligned}$$

\square

Corollary 7.11. *We have*

$$\begin{aligned}\text{ch}_t(\mathcal{R}^{\Gamma_{n-1}})_{\underline{\lambda}} &= \frac{1}{1-t} \sum_{i=0}^{l-1} t^{-i} \sum_{\substack{\underline{\mu} \uparrow \underline{\lambda}, \\ \mu^i \neq \lambda^i}} \frac{s_{\mu^i}(1, t^l, t^{2l}, \dots)}{s_{\lambda^i}(1, t^l, t^{2l}, \dots)} \\ &= \frac{1}{1-t} \sum_{i=0}^{l-1} t^{-i} \sum_{\substack{\underline{\mu} \uparrow \underline{\lambda}, \\ \mu^i \neq \lambda^i}} \frac{t^{l \cdot n(\mu^i)} H_{\lambda^i}(t^l)}{t^{l \cdot n(\lambda^i)} H_{\mu^i}(t^l)}.\end{aligned}$$

In particular, if $l = 1$ then $\text{ch}_t(\mathcal{R}^{\Gamma_{n-1}})_{\underline{\lambda}} = \frac{1}{1-t} \sum_{\mu \uparrow \lambda} \frac{s_{\mu}(1, t, t^2, \dots)}{s_{\lambda}(1, t, t^2, \dots)} = \frac{1}{1-t} \sum_{\mu \uparrow \lambda} \frac{t^{n(\mu)} H_{\lambda}(t)}{t^{n(\lambda)} H_{\mu}(t)}.$

Proof. This follows immediately from Theorem 7.6 and Theorem 7.10. \square

Part II: Quiver varieties

8 Calogero-Moser spaces

In this section we recall the construction of Calogero-Moser spaces as Nakajima quiver varieties. We also define the tautological bundle on a Calogero-Moser space and recall the Etingof-Ginzburg isomorphism.

8.1 Representations of quivers. Let Q be the cyclic quiver with l vertices and cyclic orientation. We label the vertices as $0, 1, \dots, l-1$ in such a way that there is a (unique) arrow $a_i : i \rightarrow j$ if and only if $j = i + 1 \pmod l$. Let Q_∞ be the quiver obtained from Q by adding an extra vertex, denoted ∞ , and an extra arrow $a_\infty : \infty \rightarrow 0$. We write \overline{Q}_∞ for the double quiver of Q_∞ , i.e., the quiver obtained from Q_∞ by adding, for each arrow a in Q_∞ , an arrow a^* going in the opposite direction. Let $\mathbf{d} = (d_0, \dots, d_{l-1}) \in \mathbb{Z}_{\geq 0}^l$ and let $\mathbf{d}' = (d_\infty, d_0, \dots, d_{l-1})$. We interpret \mathbf{d}' as the dimension vector for \overline{Q}_∞ so that the dimension associated to the vertex i is d_i . We fix a complex graded vector space $\mathbf{V} = \mathbf{V}_\infty \oplus \bigoplus_{i=0}^{l-1} \mathbf{V}_i$ with $\dim \mathbf{V}_i = d_i$ for each $i = \infty, 0, \dots, l-1$. Set $\widehat{\mathbf{V}} := \bigoplus_{i=0}^{l-1} \mathbf{V}_i$. Let

$$\begin{aligned} \widehat{R}(\mathbf{d}) &:= \left(\bigoplus_{i=0}^{l-1} \text{End}_{\mathbb{C}}(\mathbf{V}_i, \mathbf{V}_{i+1}) \right) \oplus \left(\bigoplus_{i=0}^{l-1} \text{End}_{\mathbb{C}}(\mathbf{V}_i, \mathbf{V}_{i-1}) \right), \\ R(\mathbf{d}') &:= \widehat{R}(\mathbf{d}) \oplus \text{End}_{\mathbb{C}}(\mathbf{V}_0, \mathbf{V}_\infty) \oplus \text{End}_{\mathbb{C}}(\mathbf{V}_\infty, \mathbf{V}_0). \end{aligned}$$

Accordingly, we denote an element of $R(\mathbf{d}')$ as $(\mathbf{X}, \mathbf{Y}, I, J) = (X_0, \dots, X_{l-1}, Y_0, \dots, Y_{l-1}, I, J)$. The algebraic group $G(\mathbf{d}) = \prod_{i=0}^{l-1} \text{GL}(\mathbf{V}_i)$ acts naturally on \mathbf{V} . It also acts on $R(\mathbf{d}')$ by change of basis. This latter action can be described explicitly as follows. If $\mathbf{g} = (g_0, \dots, g_{l-1}) \in G(\mathbf{d})$ and $(X_0, \dots, X_{l-1}, Y_0, \dots, Y_{l-1}, I, J) \in R(\mathbf{d}')$ then

$$\mathbf{g} \cdot (X_0, \dots, X_{l-1}, Y_0, \dots, Y_{l-1}, I, J) = (g_1 X_0 g_0^{-1}, \dots, g_0 X_{l-1} g_{l-1}^{-1}, g_0 Y_0 g_1^{-1}, \dots, g_{l-1} Y_{l-1} g_0^{-1}, I g_0^{-1}, g_0 J).$$

We also set $PG(\mathbf{d}) = G(\mathbf{d})/\mathbb{C}^*$, where we identify \mathbb{C}^* with the subgroup of scalar transformations. Let $\mathfrak{g}(\mathbf{d})$ be the Lie algebra of $G(\mathbf{d})$. Let

$$\mu_{\mathbf{d}} : R(\mathbf{d}') \rightarrow (\mathfrak{g}(\mathbf{d}))^* \cong \mathfrak{g}(\mathbf{d}), \quad (\mathbf{X}, \mathbf{Y}, I, J) \mapsto [\mathbf{X}, \mathbf{Y}] + JI$$

be the moment map for the $G(\mathbf{d})$ -action on $R(\mathbf{d}')$.

8.2 Quiver varieties. Let $\mathbf{d} = (d_0, \dots, d_{l-1}) \in \mathbb{Z}_{\geq 0}^l$, $\mathbf{d}' = (1, d_0, \dots, d_{l-1})$ and $\theta = (\theta_0, \dots, \theta_{l-1}) \in \mathbb{Q}^l$. Moreover, let $\text{id}_i = \text{id}_{\mathbf{V}_i}$ ($i = 0, \dots, l-1$) and $\widetilde{\theta} = (\theta_0 \text{id}_0, \theta_1 \text{id}_1, \dots, \theta_{l-1} \text{id}_{l-1})$. We define $\mathcal{X}_\theta(\mathbf{d})$ to be the affine variety

$$\mathcal{X}_\theta(\mathbf{d}) := \mu_{\mathbf{d}}^{-1}(\widetilde{\theta}) / G(\mathbf{d}) = \text{Spec } \mathbb{C}[\mu_{\mathbf{d}}^{-1}(\widetilde{\theta})]^{G(\mathbf{d})}.$$

We will always assume that the parameter θ is chosen in such a way that the affine variety $\mathcal{X}_\theta(\mathbf{d})$ is smooth. Let $\Pi_\theta := \Pi_\theta(\overline{Q}_\infty)$ be the deformed preprojective algebra with parameter θ , i.e., the quotient of the path algebra $\mathbb{C}\overline{Q}_\infty$ by the two-sided ideal generated by the element $a_\infty a_\infty^* + \sum_{i=0}^{l-1} (a_i a_i^* - a_i^* a_i) - \sum_{i=0}^{l-1} \theta_i 1_i$, where 1_i is the lazy path at vertex i . The geometric points of the scheme $\mathcal{X}_\theta(\mathbf{d})$ correspond to isomorphism classes of semisimple Π_θ -modules.

Moreover, we define $\mathcal{M}_\theta(\mathbf{d})$ to be the GIT quotient

$$\mathcal{M}_\theta(\mathbf{d}) := \mu_{\mathbf{d}}^{-1}(0) //_{\theta} G(\mathbf{d}) = \text{Proj} \bigoplus_{i \geq 0} \mathbb{C}[\mu_{\mathbf{d}}^{-1}(0)]^{\chi_{\theta}^i},$$

where $\chi_{\theta} : G(\mathbf{d}) \rightarrow \mathbb{C}^*$ is the character sending $\mathbf{g} = (g_0, \dots, g_{l-1})$ to $\prod (\det g_i)^{\theta_i}$ and $\mathbb{C}[\mu_{\mathbf{d}}^{-1}(0)]^{\chi_{\theta}^i}$ denotes the space of semi-invariant functions on $\mu_{\mathbf{d}}^{-1}(0)$, i.e., those functions f satisfying $\mathbf{g} \cdot f = \chi_{\theta}^i(\mathbf{g})f$. By definition, the space $\mathbb{C}[\mu_{\mathbf{d}}^{-1}(0)]^{\chi_{\theta}^i}$ is zero unless $i\theta \in \mathbb{Z}^l$.

The varieties $\mathcal{X}_\theta(\mathbf{d})$, $\mathcal{M}_\theta(\mathbf{d})$ can be endowed with hyper-Kähler structures as in [10, §3.6].

Notation. We will always consider the subscript i in the expressions $d_i, \mathbf{V}_i, g_i, X_i, Y_i, \theta_i$ modulo l (unless $i = \infty$).

8.3 The \mathbb{C}^* -action. The group \mathbb{C}^* acts on $R(\mathbf{d}')$ by the rule $t.(\mathbf{X}, \mathbf{Y}, I, J) = (t^{-1}\mathbf{X}, t\mathbf{Y}, I, J)$ for $t \in \mathbb{C}^*$. This action descends to an action on $\mathcal{X}_\theta(\mathbf{d})$ and $\mathcal{M}_\theta(\mathbf{d})$.

8.4 The tautological bundle on a quiver variety. Suppose that the group $G(\mathbf{d})$ acts freely on the fibre $\mu_{\mathbf{d}}^{-1}(\tilde{\theta}_{\mathbf{h}})$. Consider the trivial vector bundle $\hat{\mathcal{V}} := \mu_{\mathbf{d}}^{-1}(\tilde{\theta}) \times \hat{\mathbf{V}}$ on $\mu_{\mathbf{d}}^{-1}(\tilde{\theta})$. We regard $\hat{\mathcal{V}}$ as a \mathbb{C}^* -equivariant vector bundle by letting \mathbb{C}^* act trivially on $\hat{\mathbf{V}}$. The group $G(\mathbf{d})$ acts diagonally on $\hat{\mathcal{V}}$ according to the formula $g.(x, v) = (g.x, g^{-1}.v)$, where $g \in G(\mathbf{d})$, $x \in \mu_{\mathbf{d}}^{-1}(\tilde{\theta})$ and $v \in \hat{\mathbf{V}}$. The vector bundle $\hat{\mathcal{V}}$ descends to a \mathbb{C}^* -equivariant vector bundle $\mathcal{V} := \mu_{\mathbf{d}}^{-1}(\tilde{\theta}) \times^{G(\mathbf{d})} \hat{\mathbf{V}} = (\mu_{\mathbf{d}}^{-1}(\tilde{\theta}) \times \hat{\mathbf{V}}) // G(\mathbf{d})$ on $\mu_{\mathbf{d}}^{-1}(\tilde{\theta}) // G(\mathbf{d}) = \mathcal{X}_\theta(\mathbf{d})$, which is called the *tautological bundle*.

8.5 The Calogero-Moser space. Set $\delta = (1, \dots, 1) \in \mathbb{Z}^l$. We now fix $\mathbf{d} = n\delta$ and set

$$\theta_{\mathbf{h}} = (\theta_0, \dots, \theta_{l-1}) = (-h + H_0, H_1, \dots, H_{l-1}).$$

Since we are assuming that the parameter \mathbf{h} is generic, the group $G(\mathbf{d})$ acts freely on the fibre $\mu_{\mathbf{d}}^{-1}(\tilde{\theta}_{\mathbf{h}})$.

Definition 8.1. We define the *Calogero-Moser space* $\mathcal{N}_{\mathbf{h}}$ associated to the parameter \mathbf{h} to be the affine variety

$$\mathcal{N}_{\mathbf{h}} := \mathcal{X}_{\theta_{\mathbf{h}}}(n\delta) = \text{Spec } \mathbb{C}[\mu_{\mathbf{d}}^{-1}(\tilde{\theta}_{\mathbf{h}})]^{G(\mathbf{d})}.$$

8.6 The Etingof-Ginzburg isomorphism. We will now review the construction of the Etingof-Ginzburg map $\text{Spec } Z_{0, \mathbf{h}} \rightarrow \mathcal{N}_{\mathbf{h}}$. Throughout this section we take $\mathbf{d} = n\delta$. To simplify notation, let us drop the subscript \mathbf{d} and write μ for $\mu_{\mathbf{d}}$.

We begin by identifying $(\mathbb{C}\Gamma_n)^{\Gamma_{n-1}}$ with $\hat{\mathbf{V}} := \bigoplus_{i=0}^{l-1} \mathbf{V}_i$ as follows. The vector space $(\mathbb{C}\Gamma_n)^{\Gamma_{n-1}}$ decomposes as a direct sum $\bigoplus_{i=0}^{l-1} (\mathbb{C}\Gamma_n)_{\chi_i}^{\Gamma_{n-1}}$ of isotypic components, where $\langle \epsilon_1 \rangle \cong \mathbb{Z}/l\mathbb{Z}$ acts on $(\mathbb{C}\Gamma_n)_{\chi_i}^{\Gamma_{n-1}}$ by the character $\chi_i : \epsilon_1 \mapsto \eta^i$. We choose once and for all a linear isomorphism

$$\varpi : (\mathbb{C}\Gamma_n)^{\Gamma_{n-1}} \rightarrow \hat{\mathbf{V}} \tag{14}$$

which maps each $(\mathbb{C}\Gamma_n)_{\chi_i}^{\Gamma_{n-1}}$ onto \mathbf{V}_i . This isomorphism induces an isomorphism

$$\mathfrak{w} : \text{End}_{\mathbb{C}}((\mathbb{C}\Gamma_n)^{\Gamma_{n-1}}) \rightarrow \text{End}_{\mathbb{C}}(\hat{\mathbf{V}}). \tag{15}$$

Definition 8.2. Let

$$p : R(\mathbf{d}') \rightarrow \hat{R}(\mathbf{d}), \quad (\mathbf{X}, \mathbf{Y}, I, J) \mapsto (\mathbf{X}, \mathbf{Y})$$

be the projection. It is clearly \mathbb{C}^* -equivariant. Recall the variety $\text{Rep}_{\mathbb{C}\Gamma_n}(\mathbb{H}_{0, \mathbf{h}})$ from §3.5. Each element $\phi \in \text{Rep}_{\mathbb{C}\Gamma_n}(\mathbb{H}_{0, \mathbf{h}})$ defines endomorphisms $\phi(x_1), \phi(y_1) : \mathbb{C}\Gamma_n \rightarrow \mathbb{C}\Gamma_n$, where $x_1, y_1 \in \mathbb{H}_{0, \mathbf{h}}$. Set

$$\mathbf{X}(\phi) := \mathfrak{w} \left(\phi(x_1)|_{(\mathbb{C}\Gamma_n)^{\Gamma_{n-1}}} \right), \quad \mathbf{Y}(\phi) := \mathfrak{w} \left(\phi(y_1)|_{(\mathbb{C}\Gamma_n)^{\Gamma_{n-1}}} \right).$$

We now define a morphism of varieties

$$\Psi : \text{Rep}_{\mathbb{C}\Gamma_n}(\mathbb{H}_{0, \mathbf{h}}) \rightarrow p(\mu^{-1}(\tilde{\theta})), \quad \phi \mapsto (\mathbf{X}(\phi), \mathbf{Y}(\phi)).$$

Lemma 8.3. *The morphism Ψ is well-defined.*

Proof. The relations in $\mathbb{H}_{0,\mathbf{h}}$ imply that the operator $\phi(x_1)$ on $\mathbb{C}\Gamma_n$ commutes with the operator $\phi(e_{n-1})$. Hence $\phi(x_1)$ preserves the subspace $e_{n-1}\mathbb{C}\Gamma_n$. The same holds for $\phi(y_1)$. Therefore $\phi(x_1), \phi(y_1)$ restrict to well-defined endomorphisms $(\mathbb{C}\Gamma_n)^{\Gamma_{n-1}} \rightarrow (\mathbb{C}\Gamma_n)^{\Gamma_{n-1}}$.

If $v \in (\mathbb{C}\Gamma_n)^{\Gamma_{\chi_i^{n-1}}}$, then $\epsilon_1\phi(x_1).v = \eta\phi(x_1)\epsilon_1.v = \eta^{i+1}\phi(x_1).v$, so $x_1.v \in (\mathbb{C}\Gamma_n)^{\Gamma_{\chi_{i+1}^{n-1} \bmod l}}$. An analogous calculation shows that $y_1.v \in (\mathbb{C}\Gamma_n)^{\Gamma_{\chi_{i-1}^{n-1} \bmod l}}$, implying that $(\mathbf{X}(\phi), \mathbf{Y}(\phi)) \in \widehat{R}(\mathbf{d})$. The fact that $(\mathbf{X}(\phi), \mathbf{Y}(\phi)) \in p(\mu^{-1}(\tilde{\theta}))$ is proven in [7, Lemma 11.15]. \square

Lemma 8.4. *The morphism Ψ is \mathbb{C}^* -equivariant.*

Proof. Let $t \in \mathbb{C}^*, \phi \in \text{Rep}_{\mathbb{C}\Gamma_n}(\mathbb{H}_{0,\mathbf{h}})$ and $z \in \mathbb{H}_{0,\mathbf{h}}$. We have $(t.\phi)(z) = \phi(t^{-1}.z)$, so $(t.\phi)(x_1) = \phi(t^{-1}x_1)$ and $(t.\phi)(y_1) = \phi(ty_1)$. Hence

$$\begin{aligned} \Psi(t.\phi) &= \left(\varpi \left(t^{-1}x_1|_{(\mathbb{C}\Gamma_n)^{\Gamma_{n-1}}} \right), \varpi \left(ty_1|_{(\mathbb{C}\Gamma_n)^{\Gamma_{n-1}}} \right) \right) \\ &= (t^{-1}\mathbf{X}(\phi), t\mathbf{Y}(\phi)) = t.(\mathbf{X}(\phi), \mathbf{Y}(\phi)) \end{aligned}$$

and so the map Ψ is \mathbb{C}^* -equivariant. \square

The action of $G(\mathbf{d})$ on $\widehat{R}(\mathbf{d})$ factors through $PG(\mathbf{d})$. Moreover, the projection $G(\mathbf{d}) \twoheadrightarrow PG(\mathbf{d})$ admits a splitting $PG(\mathbf{d}) \hookrightarrow G(\mathbf{d})$ defined by the rule that the image of $PG(\mathbf{d})$ under the splitting acts trivially on $\varpi(e_n) \in \widehat{\mathbf{V}}$. This splitting allows us to define an action of $PG(\mathbf{d})$ on $\widehat{\mathbf{V}}$ and endow $p(\mu^{-1}(\tilde{\theta})) \times \widehat{\mathbf{V}}$ with the structure of a $PG(\mathbf{d})$ -equivariant vector bundle. The following theorem is a version of Theorem 11.16 and Proposition 11.24 of [7].

Theorem 8.5. *The maps $\Psi : \text{Rep}_{\mathbb{C}\Gamma_n}(\mathbb{H}_{0,\mathbf{h}}) \rightarrow p(\mu^{-1}(\tilde{\theta}))$ and $p : \mu^{-1}(\tilde{\theta}) \rightarrow p(\mu^{-1}(\tilde{\theta}))$ induce \mathbb{C}^* -equivariant isomorphisms of varieties*

$$\text{Spec } Z_{0,\mathbf{h}} \xrightarrow{\sim} p(\mu^{-1}(\tilde{\theta})) // PG(\mathbf{d}) \xleftarrow{\sim} \mathcal{N}_{\mathbf{h}} \quad (16)$$

and vector bundles

$$\mathcal{R}^{\Gamma_{n-1}} \xrightarrow{\sim} p(\mu^{-1}(\tilde{\theta})) \times^{PG(\mathbf{d})} \widehat{\mathbf{V}} \xleftarrow{\sim} \mathcal{V}. \quad (17)$$

Proof. A detailed proof of the first claim can be found in [20, Theorem 1.4]. We prove the second claim. Consider the \mathbb{C}^* -equivariant maps of trivial vector bundles of rank nl :

$$\begin{array}{ccccc} \text{Rep}_{\mathbb{C}\Gamma_n}(\mathbb{H}_{0,\mathbf{h}}) \times (\mathbb{C}\Gamma_n)^{\Gamma_{n-1}} & \xrightarrow{\Psi \times \varpi} & p(\mu^{-1}(\tilde{\theta})) \times \widehat{\mathbf{V}} & \xleftarrow{p \times \text{id}} & \mu^{-1}(\tilde{\theta}) \times \widehat{\mathbf{V}} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Rep}_{\mathbb{C}\Gamma_n}(\mathbb{H}_{0,\mathbf{h}}) & \xrightarrow{\Psi} & p(\mu^{-1}(\tilde{\theta})) & \xleftarrow{p} & \mu^{-1}(\tilde{\theta}). \end{array}$$

We have surjective group homomorphisms

$$\text{PAut}_{\Gamma_n}(\mathbb{C}\Gamma_n) \twoheadrightarrow PG(\mathbf{d}) \leftarrow G(\mathbf{d}). \quad (18)$$

The trivial vector bundle on the LHS is equivariant with respect to the $\text{PAut}_{\Gamma_n}(\mathbb{C}\Gamma_n)$ -action, the bundle in the middle is equivariant with respect to the $PG(\mathbf{d})$ -action, and the bundle on the RHS is equivariant with respect to the $G(\mathbf{d})$ -action. The vector bundle maps $\Psi \times \varpi$ and $p \times \text{id}$ intertwine these equivariant structures via the group homomorphisms (18). It follows that $\Psi \times \varpi$ and $p \times \text{id}$ descend to the following bundle maps

$$\begin{array}{ccccc} \mathcal{R}^{\Gamma_{n-1}} & \longrightarrow & p(\mu^{-1}(\tilde{\theta})) \times^{PG(\mathbf{d})} \widehat{\mathbf{V}} & \longleftarrow & \mathcal{V} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec } Z_{0,\mathbf{h}} & \xrightarrow{\sim} & p(\mu^{-1}(\tilde{\theta})) // PG(\mathbf{d}) & \xleftarrow{\sim} & \mathcal{N}_{\mathbf{h}}. \end{array}$$

Since these maps induce isomorphisms on the base spaces and on each fibre, they are isomorphisms. The fact that these isomorphisms are \mathbb{C}^* -equivariant follows from the equivariance of the original bundle maps $\Psi \times \varpi$ and $p \times \text{id}$.

□

Definition 8.6. We call the composite isomorphism $\text{Spec } Z_{0,\mathbf{h}} \xrightarrow{\sim} \mathcal{N}_{\mathbf{h}}$ from (16) the *Etingof-Ginzburg isomorphism* and denote it with the symbol EG.

9 Combinatorics II

The main goal of this section is to develop the vocabulary required to talk about bead diagrams (abaci). We explain how one can read off the core and quotient of a partition from a bead diagram. We also recall the classical bijection between partitions of nl with a trivial l -core and l -multipartitions of n . These considerations will play an important role in the subsequent sections, where we match the \mathbb{C}^* -fixed points in the Calogero-Moser space with the fixed points in $\text{Spec } Z_{0,\mathbf{h}}$.

9.1 Bead diagrams. We call an element (i, j) of $\mathbb{Z}_{\leq -1} \times \{0, \dots, l-1\}$ a *point*. According to the natural geometric intuition we say that the point (i, j) lies to the left of (i, j') if $j < j'$, and that (i, j) lies above (i', j) if $i' < i$.

A *bead diagram* is a function $f : \mathbb{Z}_{\leq -1} \times \{0, \dots, l-1\} \rightarrow \{0, 1\}$ which takes value 1 for only finitely many points. If $f(i, j) = 1$ we say that the point (i, j) is occupied by a *bead*. If $f(i, j) = 0$ we say that the point (i, j) is *empty*. Suppose that a point (i, j) is empty and that there exists an $i' < i$ such that the point (i', j) is occupied by a bead. Then we call the point (i, j) a *gap*.

We say that a point $(i, j) \in \mathbb{Z}_{\leq -1} \times \{0, \dots, l-1\}$ is in the $(-i)$ -th row and j -th column (or runner) of the bead diagram. We call the i -th row full (empty) if every point (i, k) for $k = 0, \dots, l-1$ is occupied by a bead (is empty). A row is called redundant if it is a full row and if all the rows above it are full.

Definition 9.1. Let $\mu = (\mu_1, \mu_2, \dots)$ be a partition of any integer such that $\mu_s \neq 0$ but $\mu_{s+1} = 0$. Let $p \geq s$. Set

$$\beta_i^p = \mu_i + p - i \quad (1 \leq i \leq p).$$

We call $\{\beta_i^p \mid 1 \leq i \leq p\}$ a set of β -numbers for μ . Note that $|\{\beta_i^p \mid 1 \leq i \leq p\}| = p$. It is clear that from a set of β -numbers one can uniquely recover the corresponding partition μ .

Definition 9.2. Given a set of β -numbers $\{\beta_i^p \mid 1 \leq i \leq p\}$ we can naturally associate to it a bead diagram by the rule

$$f(i, j) = 1 \iff -(i+1) \cdot l + j \in \{\beta_i^p \mid 1 \leq i \leq p\}.$$

If p is the smallest multiple of l satisfying $p \geq s$ we denote the resulting bead diagram with $\mathbb{B}(\mu)$. The diagram $\mathbb{B}(\mu)$ has no redundant rows and the number of beads in $\mathbb{B}(\mu)$ is a multiple of l .

Remark 9.3. Conversely, if we are given a bead diagram f , the set $\{-(i+1) \cdot l + j \mid f(i, j) = 1\}$ is a set of β -numbers for some partition. The relationship between bead diagrams, sets of β -numbers and partitions can therefore be illustrated as follows

$$\{\text{bead diagrams}\} \longleftrightarrow \{\text{sets of } \beta\text{-numbers}\} \twoheadrightarrow \{\text{partitions}\},$$

where the set of partitions contains partitions of an arbitrary integer.

To simplify the graphical presentation, we will truncate all the bead diagrams by removing all the empty rows at the bottom.

9.2 Cores and quotients. Let $f : \mathbb{Z}_{\leq -1} \times \{0, \dots, l-1\} \rightarrow \{0, 1\}$ be a bead diagram. Suppose that the point (i, j) with $i < -1$ is occupied by a bead, i.e., $f(i, j) = 1$, and that $f(i+1, j) = 0$. To *slide* or *move* the bead in position (i, j) *upward* means to modify the function f by setting $f'(i, j) = 0, f'(i+1, j) = 1$ and $f' = f$ otherwise.

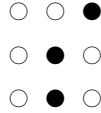
Definition 9.4. Let μ be a partition of an integer k . Take any bead diagram f corresponding to μ . We obtain a new bead diagram f' by sliding beads upward as long as it is possible. We call the partition corresponding to the bead diagram f' the l -core of μ , denoted $\text{Core}(\mu)$. Let $\heartsuit(l)$ denote the set of all l -cores. If the l -core of μ is the empty partition, we say that μ has a trivial l -core. We denote the set of partitions of k with trivial l -core by $\mathcal{P}_{\emptyset}(k)$. More generally, if $\nu \in \heartsuit(l)$, we set

$$\mathcal{P}_{\nu}(k) = \{\mu \in \mathcal{P}(k) \mid \text{Core}(\mu) = \nu\}.$$

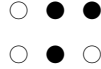
Definition 9.5. Now consider the bead diagram $\mathbb{B}(\mu)$. Each column of $\mathbb{B}(\mu)$ can itself be considered as a bead diagram for $l = 1$. Let $Q^i(\mu)$ denote the partition corresponding to the i -th column. We call the multipartition $\underline{\text{Quot}}(\mu) := (Q^0(\mu), Q^1(\mu), \dots, Q^{l-1}(\mu))$ the l -quotient of μ .

A partition is determined uniquely by its l -core and l -quotient ([13, Theorem 2.7.30]).

Example 9.6. Consider the partition $\mu = (5, 3, 2)$ and take $l = 3$. The first-column hook-lengths are 2, 4, 7. They form a set of β -numbers. The corresponding bead diagram $\mathbb{B}(\mu)$ is



We see that $\underline{\text{Quot}}(\mu) = (\emptyset, (1, 1), \emptyset)$. After sliding all the beads upward we obtain the bead diagram



corresponding to the β -numbers 1, 2, 4. Hence the 3-core of μ is the partition $(2, 1, 1)$.

9.3 Rim-hooks. The *rim* of $\mathbb{Y}(\mu)$ is the subset of $\mathbb{Y}(\mu)$ consisting of the cells (i, j) such that $(i+1, j+1)$ does not lie in $\mathbb{Y}(\mu)$. Fix a cell $(i, j) \in \mathbb{Y}(\mu)$. Recall that by the hook associated to (i, j) we mean the subset of $\mathbb{Y}(\mu)$ consisting of all the cells (i, k) with $k \geq j$ and all the cells (k, j) with $k \geq i$. We define the *rim-hook* associated to the cell (i, j) to be the intersection of the set $\{(i', j') \mid i' \geq i, j' \geq j\}$ with the rim of $\mathbb{Y}(\mu)$. We call a rim-hook an l -rim-hook if it contains l cells.

The l -core of μ can also be characterised as the subpartition μ' of μ obtained from μ by a successive removal of l -rim-hooks, in whichever order (see [13, Theorem 2.7.16]). We recall the following well-known lemma.

Lemma 9.7. Let R be an l -rim-hook in μ and set $\mu' := \mu - R$. Then $\underline{\text{Quot}}(\mu') = \underline{\text{Quot}}(\mu) - \square$ for some $\square \in \underline{\text{Quot}}(\mu)$.

Proof. This follows directly from [13, Lemma 2.7.13]. □

9.4 From partitions to multipartitions. Now suppose that $\mu \in \mathcal{P}_{\emptyset}(nl)$. Lemma 9.7 implies that $\underline{\text{Quot}}(\mu) \in \mathcal{P}(l, n)$. Since a partition with trivial core is uniquely determined by its quotient, we conclude that there exists a bijection

$$\mathcal{P}_{\emptyset}(nl) \rightarrow \mathcal{P}(l, n), \quad \mu \mapsto \underline{\text{Quot}}(\mu).$$

10 Reflection functors and Hilbert schemes

Assume in this section that $l > 1$. We are going to recall Nakajima reflection functors and explain the diffeomorphism between the Calogero-Moser space and a certain subscheme of a Hilbert scheme.

10.1 The \tilde{S}_l -action on the parameter space. In this section all subscripts should be regarded modulo l . Let \tilde{S}_l denote the affine symmetric group. It has a Coxeter presentation with generators $\sigma_0, \dots, \sigma_{l-1}$ and relations

$$\sigma_i^2 = 1, \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad (0 \leq i \leq l-1).$$

It acts naturally on the parameter space \mathbb{Q}^l by the rule $\sigma_i \cdot \theta = \theta'$, where

$$\theta'_i = -\theta_i, \quad \theta'_{i-1} = \theta_{i-1} + \theta_i, \quad \theta_{i+1} = \theta_{i+1} + \theta_i, \quad \theta'_j = \theta_j \quad (j \notin \{i-1, i, i+1\})$$

for all $0 \leq i \leq l-1$. More details about this action can be found in §15.10.

10.2 The \tilde{S}_l -action on partitions. We will now define the action of \tilde{S}_l on the set of all partitions and state some of its properties. We will later use this action to describe the behaviour of the \mathbb{C}^* -fixed points under reflection functors. Recall that $\mathcal{P} := \bigsqcup_{m \in \mathbb{Z}_{\geq 0}} \mathcal{P}(m)$ and $\underline{\mathcal{P}} := \bigsqcup_{m \in \mathbb{Z}_{\geq 0}} \mathcal{P}(l, m)$. We will need the following definition, reminiscent of the combinatorics of the Fock space.

Definition 10.1. Let $k \in \{0, \dots, l-1\}$. Consider the Young diagram $\mathbb{Y}(\mu)$ as a subset of the $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ space. We say that a cell $(i, j) \in \mathbb{Y}(\mu)$ is *removable* relative to μ if $\mathbb{Y}(\mu) - \{(i, j)\}$ is the Young diagram of a partition. We say that it is *k-removable* relative to μ if additionally $c(i, j) = j - i = k \bmod l$. We call a cell $(i, j) \notin \mathbb{Y}(\mu)$ *addable* relative to μ if $\mathbb{Y}(\mu) \cup \{(i, j)\}$ is the Young diagram of a partition. We call it *k-addable* relative to μ if additionally $c(i, j) = j - i = k \bmod l$.

We develop the combinatorics of removability and addability in more detail in §15.6. We will, in particular, require Lemma 15.13 proven there.

Definition 10.2. Suppose that $\mu \in \mathcal{P}$. Let $\mathbf{T}_k(\mu)$ be the partition such that

$$\mathbb{Y}(\mathbf{T}_k(\mu)) = \mathbb{Y}(\mu) \cup \{\square \text{ is } k\text{-addable relative to } \mu\} - \{\square \text{ is } k\text{-removable relative to } \mu\}. \quad (19)$$

The group \tilde{S}_l acts on \mathcal{P} by the rule

$$\sigma_i * \lambda = \mathbf{T}_i(\lambda) \quad (\lambda \in \mathcal{P}, i \in \mathbb{Z}/l\mathbb{Z}).$$

This action was defined in [18, §4]. It also plays a role in the combinatorics of the Schubert calculus of the affine Grassmannian, see [16, §8.2] and [17, §11]. Recall that $\heartsuit(l) \subset \mathcal{P}$ denotes the set of all l -cores and \emptyset denotes the empty partition. By [17, Proposition 22], we have $\tilde{S}_l * \emptyset = \heartsuit(l)$.

We will now recall how the \tilde{S}_l -action behaves with respect to cores and quotients. Let us identify the finite symmetric group S_l with the group of permutations of the set $\{0, \dots, l-1\}$. Let $s_i \in S_l$ ($i = 1, \dots, l-1$) be the simple transposition swapping $i-1$ and i . Let $\underline{\lambda} = (\lambda^0, \dots, \lambda^{l-1}) \in \underline{\mathcal{P}}$. The group S_l acts on $\underline{\mathcal{P}}$ by the rule

$$w \cdot \underline{\lambda} = (\lambda^{w(0)}, \dots, \lambda^{w(l-1)}), \quad w \in S_l.$$

In particular, $s_i \cdot \underline{\lambda}$ is the multipartition obtained from $\underline{\lambda}$ by swapping λ^{i-1} and λ^i . Recall the group homomorphism

$$\text{pr} : \tilde{S}_l \twoheadrightarrow S_l, \quad \sigma_i \mapsto s_i \quad (i = 1, \dots, l-1), \quad \sigma_0 \mapsto s_0,$$

where s_0 is the transposition swapping 0 and $l-1$.

Proposition 10.3. Let $\mu \in \mathcal{P}$ and $\sigma \in \tilde{S}_l$. Then

$$\text{Core}(\sigma * \mu) = \sigma * \text{Core}(\mu), \quad \underline{\text{Quot}}(\sigma * \mu) = \text{pr}(\sigma) \cdot \underline{\text{Quot}}(\mu).$$

Proof. See [18, Proposition 4.1.3]. □

10.3 Partitions and the cyclic quiver. Let $N_i(\lambda)$ be the number of cells of l -content i in $\mathbb{Y}(\lambda)$. Using this notation, the l -residue of λ equals $\sum_{i=0}^{l-1} N_i(\lambda)t^i$. Consider the map

$$\mathfrak{d} : \mathcal{P} \rightarrow \mathbb{Z}^l, \quad \lambda \mapsto \mathbf{d}_\lambda := (N_0(\lambda), \dots, N_{l-1}(\lambda)). \quad (20)$$

We interpret this map as assigning to every partition a dimension vector for the cyclic quiver with l vertices. Let

$$\mathbb{Z}_\heartsuit = \{\mathbf{d} \in (\mathbb{Z}_{\geq 0})^l \mid \mathbf{d} = \mathbf{d}_\nu \text{ for some } \nu \in \heartsuit(l)\}$$

be the set of all dimension vectors corresponding to l -cores. By [13, Theorem 2.7.41] an l -core is determined uniquely by its l -residue. Hence (20) induces a bijection

$$\mathfrak{d} : \heartsuit(l) \longleftrightarrow \mathbb{Z}_\heartsuit, \quad \nu \mapsto \mathbf{d}_\nu.$$

There is also an action of \tilde{S}_l on \mathbb{Z}^l defined as follows. Let $\mathbf{d} = (d_0, \dots, d_{l-1}) \in \mathbb{Z}^l$. Then $\sigma_i * \mathbf{d} = \mathbf{d}' := (d'_0, \dots, d'_{l-1})$, where $d'_j = d_j$ ($j \neq i$) and

$$d'_i = d_{i+1} + d_{i-1} - d_i \quad (i \neq 0), \quad d'_0 = d_1 + d_{l-1} - d_0 + 1 \quad (i = 0).$$

The following proposition follows by an elementary calculation from Lemma 15.13.

Proposition 10.4. *The following diagram is \tilde{S}_l -equivariant*

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\mathfrak{d}} & \mathbb{Z}^l \\ \uparrow & & \uparrow \\ \heartsuit(l) & \xrightarrow[\mathfrak{d}]{\sim} & \mathbb{Z}_\heartsuit \end{array}$$

Set $\delta = (1, \dots, 1) \in \mathbb{Z}^l$. Let $\sigma_i \in \tilde{S}_l$ and $\nu \in \heartsuit(l)$. Then $\sigma_i * (n\delta + \mathbf{d}_\nu) = n\delta + \sigma_i * \mathbf{d}_\nu$ and $\sigma_i * \mathbf{d}_\nu = \mathfrak{d}(\sigma_i * \nu)$. By [13, Theorem 2.7.41] any partition λ of $nl + |\sigma_i * \nu|$ such that $\mathfrak{d}(\lambda) = n\delta + \sigma_i * \mathbf{d}_\nu$ has l -core $\sigma_i * \nu$. Hence

$$\mathcal{P}_{\sigma_i * \nu}(nl + |\sigma_i * \nu|) = \mathfrak{d}^{-1}(n\delta + \sigma_i * \mathbf{d}_\nu).$$

10.4 Reflection functors. Fix $i \in \{0, \dots, l-1\}$. Let $\theta = (\theta_0, \dots, \theta_{l-1}) \in \mathbb{Q}^l$ be such that $\theta_i \neq 0$. Choose $\sigma_i \in \tilde{S}_l$ and $\nu \in \heartsuit(l)$. Let

$$\mathfrak{R}_i : \mathcal{X}_\theta(n\delta + \mathbf{d}_\nu) \rightarrow \mathcal{X}_{\sigma_i \cdot \theta}(n\delta + \sigma_i * \mathbf{d}_\nu) \quad (21)$$

be the reflection functor associated to the simple reflection σ_i . These functors were defined in [3, §2], [4, §5] by Crawley-Boevey using the language of quiver representations and in [23, §3] by Nakajima using the language of hyper-Kähler manifolds. These two descriptions are equivalent, as shown in Proposition 4.19 in [23]. In Nakajima's framework, one can endow the varieties $\mathcal{X}_\theta(n\delta + \mathbf{d}_\nu), \mathcal{X}_{\sigma_i \cdot \theta}(n\delta + \sigma_i * \mathbf{d}_\nu)$ with hyper-Kähler structures which make the map \mathfrak{R}_i a $U(1)$ -equivariant hyper-Kähler isometry.

Let us also use the symbol \mathfrak{R}_i to refer to the map $\mu_{n\delta + \mathbf{d}_\nu}^{-1}(\tilde{\theta}) \rightarrow \mu_{n\delta + \sigma_i * \mathbf{d}_\nu}^{-1}(\widetilde{\sigma \cdot \theta})$ lifting (21). We now briefly recall the quiver-theoretic description of the reflection functors, which we will later use in our calculations. To simplify notation, set $\mathbf{d} := \mathbf{d}_\nu$ and $\mathbf{d}' := \sigma_i * \mathbf{d}_\nu$. Let us fix $\mathbb{Z}/l\mathbb{Z}$ -graded complex vector spaces $\widehat{\mathbf{V}}^\nu := \bigoplus_{j=0}^{l-1} \mathbf{V}_j^\nu$, where $\dim_{\mathbb{C}} \mathbf{V}_j^\nu = n + d_j$, and $\widehat{\mathbf{V}}^{\sigma_i * \nu} := \bigoplus_{j=0}^{l-1} \mathbf{V}_j^{\sigma_i * \nu}$, where $\dim_{\mathbb{C}} \mathbf{V}_j^{\sigma_i * \nu} = n + d'_j$. Set $\mathbf{V}^\nu = \widehat{\mathbf{V}}^\nu \oplus \mathbf{V}_\infty$ and $\mathbf{V}^{\sigma_i * \nu} = \widehat{\mathbf{V}}^{\sigma_i * \nu} \oplus \mathbf{V}_\infty$, where $\dim_{\mathbb{C}} \mathbf{V}_\infty = 1$.

Let $\rho = (X_0, \dots, X_{l-1}, Y_0, \dots, Y_{l-1}, I, J) \in \mu_{n\delta + \mathbf{d}}^{-1}(\tilde{\theta})$. It is a representation of the quiver $\overline{\mathcal{Q}}_\infty$ with underlying vector space \mathbf{V}^ν . The reflected quiver representation

$$\mathfrak{R}_i(\rho) := (X'_0, \dots, X'_{l-1}, Y'_0, \dots, Y'_{l-1}, I', J')$$

is defined as follows. Suppose that $i \neq 0$. We have maps

$$\mathbf{V}_i^\nu \xrightarrow{Y_i - X_i} \mathbf{V}_{i-1}^\nu \oplus \mathbf{V}_{i+1}^\nu \xrightarrow{X_{i-1} + Y_{i+1}} \mathbf{V}_i^\nu. \quad (22)$$

Set $\psi := Y_i - X_i$ and $\phi := X_{i-1} + Y_{i+1}$. The preprojective relations ensure that we have a splitting $\mathbf{V}_{i-1}^\nu \oplus \mathbf{V}_{i+1}^\nu = \text{Im } \psi \oplus \ker \phi$. The underlying vector space of the quiver representation $\mathfrak{R}_i(\rho)$ is obtained from \mathbf{V}^ν by replacing \mathbf{V}_i with $\ker \phi$. By the definition of the action of σ_i on dimension vectors we have an isomorphism of vector spaces $\mathbf{V}^{\sigma_i * \nu} \cong \ker \phi \oplus \bigoplus_{j \neq i} \mathbf{V}_j^\nu \oplus \mathbf{V}_\infty$ preserving the quiver grading. We now define the linear maps which constitute $\mathfrak{R}_i(\rho)$. We have $X'_j = X_j$ unless $j \in \{i-1, i\}$. We also have $Y'_j = Y_j$ unless $j \in \{i, i+1\}$. Set $I' = I$ and $J' = J$. The maps X'_i and Y'_i are defined as the composite maps

$$\begin{aligned} X'_i : \ker \phi &\hookrightarrow \ker \phi \oplus \text{Im } \psi = \mathbf{V}_{i-1}^\nu \oplus \mathbf{V}_{i+1}^\nu \twoheadrightarrow \mathbf{V}_{i+1}^\nu, \\ Y'_i : \ker \phi &\hookrightarrow \ker \phi \oplus \text{Im } \psi = \mathbf{V}_{i-1}^\nu \oplus \mathbf{V}_{i+1}^\nu \twoheadrightarrow \mathbf{V}_{i-1}^\nu. \end{aligned}$$

The maps X'_{i-1} and Y'_{i+1} are defined as the composite maps

$$\begin{aligned} X'_{i-1} : \mathbf{V}_{i-1}^\nu &\xrightarrow{\cdot(-\theta_i)} \mathbf{V}_{i-1}^\nu \hookrightarrow \mathbf{V}_{i-1}^\nu \oplus \mathbf{V}_{i+1}^\nu = \ker \phi \oplus \text{Im } \psi \twoheadrightarrow \ker \phi, \\ Y'_{i+1} : \mathbf{V}_{i+1}^\nu &\xrightarrow{\cdot(-\theta_i)} \mathbf{V}_{i+1}^\nu \hookrightarrow \mathbf{V}_{i-1}^\nu \oplus \mathbf{V}_{i+1}^\nu = \ker \phi \oplus \text{Im } \psi \twoheadrightarrow \ker \phi \xrightarrow{\cdot(-1)} \ker \phi. \end{aligned}$$

The minus signs before X_i in (22) as well as in the last arrow above come from the fact that our quiver does not have a sink at the vertex i as in [4, §5] - hence the need for these adjustments. We can in fact describe the maps X'_{i-1} and Y'_{i+1} by some very explicit formulas. Let $v \in \mathbf{V}_{i-1}^\nu$ and $u \in \mathbf{V}_{i+1}^\nu$. Then

$$X'_{i-1}(v) = -\theta_i v + Y_i X_{i-1}(v) - X_i X_{i-1}(v), \quad Y'_{i+1}(u) = \theta_i u + X_i Y_{i+1}(u) - Y_i Y_{i+1}(u).$$

The maps X'_i and Y'_i can also be described rather explicitly. Suppose that $v \in \ker \phi$. Then $v = u + w$ for some uniquely determined $u \in \mathbf{V}_{i-1}^\nu$ and $w \in \mathbf{V}_{i+1}^\nu$ such that $X_{i-1}(u) = -Y_{i+1}(w)$. Then $X'_i(v) = w$ and $Y'_i(v) = u$. If $i = 0$ the definition of $\mathfrak{R}_i(\rho)$ is analogous, although slightly more complicated because one needs to take account of the presence of a third neighbouring vertex.

10.5 The Hilbert scheme. Let K be a positive integer. We let $\text{Hilb}(K)$ denote the *Hilbert scheme* of K points in \mathbb{C}^2 . The underlying set of the scheme $\text{Hilb}(K)$ consists of ideals of $\mathbb{C}[z_1, z_2]$ of colength K , i.e., ideals $I \subset \mathbb{C}[z_1, z_2]$ such that $\dim \mathbb{C}[z_1, z_2]/I = K$.

We now briefly review the construction of the scheme $\text{Hilb}(K)$ as a Nakajima quiver variety. Let Q^J denote the Jordan quiver, i.e., the quiver with one vertex 0 and a single loop. Let Q_∞^J denote the extension of the Jordan quiver by a vertex ∞ and an arrow $\infty \rightarrow 0$. Finally let \overline{Q}_∞^J denote the double of this quiver. Consider the space $R(K) = \{(X, Y : \mathbb{C}^K \rightarrow \mathbb{C}^K, A : \mathbb{C} \rightarrow \mathbb{C}^K, B : \mathbb{C}^K \rightarrow \mathbb{C})\}$ of representations of this quiver with dimension vector $(d_\infty = 1, d_0 = K)$. The algebraic group $\text{GL}(K)$ acts naturally on $R(K)$ by conjugation. We have the following moment map

$$\mu_K : R(K) \rightarrow \mathfrak{gl}(K), \quad (X, Y, A, B) \mapsto [X, Y] + AB$$

for this action. Consider the GIT quotient $\mu_K^{-1}(0)/_{-1}\text{GL}(K)$. The stability condition forces $B = 0$. By [22, Theorem 1.14] there exists an isomorphism

$$\text{Hilb}(K) \xrightarrow{\cong} \mu_K^{-1}(0)/_{-1}\text{GL}(K) \tag{23}$$

sending an ideal I to the quadruple $(X_I, Y_I, A_I, 0)$, where $X_I \in \text{End}(\mathbb{C}[z_1, z_2]/I)$ is multiplication by $z_1 \bmod I$, $Y_I \in \text{End}(\mathbb{C}[z_1, z_2]/I)$ is multiplication by $z_2 \bmod I$ and $A_I \in \text{Hom}(\mathbb{C}, \mathbb{C}[z_1, z_2]/I)$ is defined by $A_I(1) = 1 \bmod I$. The inverse of this isomorphism sends a quadruple $(X, Y, A, 0)$ to the kernel of the map $\phi : \mathbb{C}[z_1, z_2] \rightarrow \mathbb{C}^K$ defined by $\phi(f) = f(X, Y)A(1)$.

We let \mathbb{C}^* act on $\mu_K^{-1}(0)/_{-1}\text{GL}(K)$ by the rule $t.(X, Y, A, 0) = (t^{-1}X, tY, A, 0)$. We also let \mathbb{C}^* act on $\mathbb{C}[z_1, z_2]$ by the rule $t.z_1 = tz_1, t.z_2 = t^{-1}z_2$. This action induces an action on $\text{Hilb}(K)$. The isomorphism (23) is \mathbb{C}^* -equivariant with respect to these actions. The \mathbb{C}^* -fixed points in $\text{Hilb}(K)$ are precisely the monomial ideals in $\mathbb{C}[z_1, z_2]$, i.e., the ideals generated by monomials. Let

$\lambda \in \mathcal{P}(K)$. Let I_λ be the \mathbb{C} -span of the monomials $\{z_1^i z_2^j \mid (i+1, j+1) \notin \mathbb{Y}(\lambda)\}$. We have a bijection

$$\mathcal{P}(K) \longleftrightarrow \text{Hilb}(K)^{\mathbb{C}^*}, \quad \lambda \mapsto I_\lambda.$$

. The \mathbb{C}^* -action on $\mathbb{C}[z_1, z_2]$ gives rise to a \mathbb{C}^* -module structure on the vector space $\mathbb{C}[z_1, z_2]/I_\lambda$. Let $\mathcal{T}(K)$ denote the tautological bundle on $\text{Hilb}(K)$. Its fibre at I is isomorphic to $\mathbb{C}[z_1, z_2]/I$.

Lemma 10.5. *The \mathbb{C}^* -character of $\mathbb{C}[z_1, z_2]/I_\lambda$ is equal to $\text{Res}_{\lambda^t}(t)$. In particular, it follows that $\text{ch}_t \mathcal{T}(K)_{I_\lambda} = \text{Res}_{\lambda^t}(t)$.*

Proof. This is immediate from the definition of I_λ and the \mathbb{C}^* -action on $\mathbb{C}[z_1, z_2]$. \square

There is also a $\mathbb{Z}/l\mathbb{Z}$ -action on $\text{Hilb}(K)$ induced by the $\mathbb{Z}/l\mathbb{Z}$ -action on $\mathbb{C}[z_1, z_2]$ given by $\epsilon.z_1 = \eta^{-1}z_1, \epsilon.z_2 = \eta z_2$. The isomorphism (23) is equivariant with respect to the $\mathbb{Z}/l\mathbb{Z}$ -action.

10.6 From the Calogero-Moser space to the Hilbert scheme. Set $-\mathbf{1} := (-1/l, \dots, -1/l) \in \mathbb{Q}^l$ and $-\frac{1}{2} := (-1/2l, \dots, -1/2l) \in \mathbb{Q}^l$. Let $w \in \tilde{S}_l$ and set $\theta := w^{-1} \cdot (-\frac{1}{2}) \in \mathbb{Q}^l$ as well as $\gamma := w * n\delta \in \mathbb{Z}^l$. We have $\gamma = n\delta + \gamma_0$, where $\gamma_0 = w * \emptyset$. Let $\nu := \mathfrak{d}^{-1}(\gamma_0)$ be the l -core corresponding to γ_0 . We choose a reduced expression $w = \sigma_{i_1} \dots \sigma_{i_m}$ for w in \tilde{S}_l . Composing reflection functors yields a $U(1)$ -equivariant hyper-Kähler isometry

$$\mathfrak{R}_{i_1} \circ \dots \circ \mathfrak{R}_{i_m} : \mathcal{X}_\theta(n\delta) \rightarrow \mathcal{X}_{-\frac{1}{2}}(\gamma). \quad (24)$$

By [10, §3.7] there exists a $U(1)$ -equivariant diffeomorphism

$$\mathcal{X}_{-\frac{1}{2}}(\gamma) \rightarrow \mathcal{M}_{-\mathbf{1}}(\gamma). \quad (25)$$

Set $K = nl + |\nu|$. By forgetting the $\mathbb{Z}/l\mathbb{Z}$ -grading we obtain an embedding

$$\mathcal{M}_{-\mathbf{1}}(\gamma) \hookrightarrow \mu_K^{-1}(0) //_{-1} \text{GL}(K) \xrightarrow{\cong} \text{Hilb}(K). \quad (26)$$

We now describe the image of this embedding. By [10, Lemma 7.8] there is a component $\text{Hilb}(\nu)$ of $\text{Hilb}(K)^{\mathbb{Z}/l\mathbb{Z}}$ whose generic points have the form $V(I_\nu) \cup O$, where O is a union of n distinct free $\mathbb{Z}/l\mathbb{Z}$ -orbits in \mathbb{C}^2 . Moreover, the embedding (26) restricts to a $U(1)$ -equivariant hyper-Kähler isometry $\mathcal{M}_{-\mathbf{1}}(\gamma) \xrightarrow{\cong} \text{Hilb}(\nu)$. We note that $\text{Hilb}(\nu)^{\mathbb{C}^*} = \{I_\lambda \mid \lambda \in \mathcal{P}_\nu(nl + |\nu|)\}$. Finally, let

$$\Phi : \mathcal{X}_{-\frac{1}{2}}(\gamma) \rightarrow \mathcal{M}_{-\mathbf{1}}(\gamma) \rightarrow \text{Hilb}(\nu) \quad (27)$$

be the composition of (25) and (26).

11 Matching the \mathbb{C}^* -fixed points

11.1 The problem. Let θ and γ be as in §10.6. Set $\mathbf{h} := (h, H_1, \dots, H_{l-1})$, where $H_j = \theta_j$ ($1 \leq j \leq l-1$) and $h = -\theta_0 - \sum_{j=1}^{l-1} H_j$. With this choice of \mathbf{h} the variety $\text{Spec } Z_{0, \mathbf{h}}$ is smooth. Composing the Etingof-Ginzburg map with (24) and (27) we obtain a $U(1)$ -equivariant (non-algebraic) isomorphism

$$\text{Spec } Z_{0, \mathbf{h}} \xrightarrow{\text{EG}} \mathcal{X}_\theta(n\delta) \xrightarrow{\mathfrak{R}_{i_1} \circ \dots \circ \mathfrak{R}_{i_m}} \mathcal{X}_{-\frac{1}{2}}(\gamma) \xrightarrow{\Phi} \text{Hilb}(\nu). \quad (28)$$

The isomorphism (28) induces a bijection between the labelling sets of the \mathbb{C}^* -fixed points

$$\Omega : \mathcal{P}(l, n) \rightarrow \mathcal{P}_\nu(nl + |\nu|). \quad (29)$$

It is natural to ask the following question.

Question. How to describe the bijection (29) explicitly?

11.2 A counterexample. This problem was already considered by Gordon in [10]. His approach relies on the following function. The c -function $c : \mathcal{P}(l, n) \times \mathbb{Q}^l \rightarrow \mathbb{Q}$ is defined by the formula

$$c_{\mathbf{h}}(\underline{\lambda}) = l \sum_{i=1}^{l-1} |\lambda^i| (H_1 + \dots + H_i) - l \left(\frac{n(n-1)}{2} + \sum_{i=0}^{l-1} n(\lambda^i) - n((\lambda^i)^t) \right) h.$$

In particular, if $l = 1$ then

$$c_h(\lambda) = - \left(\frac{n(n-1)}{2} + n(\lambda) - n(\lambda^t) \right) h. \quad (30)$$

Given $\mathbf{h} \in \mathbb{Q}$, the c -function induces an ordering on $\mathcal{P}(l, n)$ given by the rule

$$\underline{\mu} <_{\mathbf{h}} \underline{\lambda} \iff c_{\mathbf{h}}(\underline{\mu}) < c_{\mathbf{h}}(\underline{\lambda}).$$

Call this ordering the c -order. Dependence on this order decomposes the parameter space \mathbb{Q}^l into a finite number of so-called c -chambers. Gordon claims in [10, §2.5] that the c -order is total inside c -chambers.

This claim is however false. In fact, it is easy to obtain counterexamples in which the c -order is not total for all values of \mathbf{h} . For example, take $l = 1$. It follows immediately from (30) that $c_h(\lambda) = c_h(\mu)$ for all values of h if λ and μ are two symmetric partitions in $\mathcal{P}(n)$. There are other examples. Take $\mu = (6, 3, 2, 2, 2)$. Then $n(\mu) = 3 + 4 + 6 + 8 = 21$. Since $\mu^t = (5, 5, 2, 1, 1, 1)$ we have $n(\mu^t) = 5 + 4 + 3 + 4 + 5 = 21$. It follows that μ and μ^t are incomparable in the c -order for all values of h .

Let us now recall the role the c -function plays in Gordon's proof. One can import the labelling of the \mathbb{C}^* -fixed points from $\text{Spec } Z_{0, \mathbf{h}}$ to $\mathcal{M}_{2\theta}(n\delta)$ through the composition of the Etingof-Ginzburg map and the rotation of the complex structure on $\mathcal{X}_{\theta}(n\delta)$. Let us use the symbol $m_{\underline{\lambda}}$ to denote the fixed point labelled by $\underline{\lambda}$. The $U(1)$ -action on $\mathcal{M}_{2\theta}(n\delta)$ is Hamiltonian and gives rise to a moment map $\mu_{U(1)} : \mathcal{M}_{2\theta}(n\delta) \rightarrow (\text{Lie } U(1))^*$. Evaluating this moment map at $-2\sqrt{-1}$ gives rise to a Morse function

$$f_{2\theta} : \mathcal{M}_{2\theta}(n\delta) \rightarrow \mathbb{R}, \quad x \mapsto (\mu_{U(1)}(x))(-2\sqrt{-1}).$$

One can define a Morse function $f_{\text{Hilb}} : \text{Hilb}(K) \rightarrow \mathbb{R}$ in an analogous fashion. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{2\theta}(n\delta) & \xrightarrow{\sim} & \text{Hilb}(\nu) \\ & \searrow f_{2\theta} & \swarrow f_{\text{Hilb}} \\ & \mathbb{R} & \end{array} \quad (31)$$

By [10, Lemma 5.3], $f_{2\theta}(m_{\underline{\lambda}}) = c_{\mathbf{h}}(\underline{\lambda})$. Gordon defines a certain bijection $\tau_{\mathbf{h}} : \mathcal{P}(l, n) \rightarrow \mathcal{P}_{\nu}(nl + |\nu|)$ which is a modification of the classical correspondence between partitions with l -core ν and their l -quotients. He shows that $f_{2\theta}(m_{\underline{\lambda}}) = f_{\text{Hilb}}(I_{\tau(\underline{\lambda}^t)})$. Assuming that all values of $c_{\mathbf{h}}(\underline{\lambda})$ are distinct, he concludes that $\tau(\underline{\lambda}^t) = \Omega(\underline{\lambda})$. However, the distinctness assumption is false, so the proof is incomplete.

11.3 Strategy. We will show that the bijection (29) is indeed given by a version of the l -quotient map. The proof of this fact is rather complicated. We will split the problem into several parts. We first consider varieties of the form $\mathcal{X}_{\phi}(n\delta + \mathbf{d}_{\rho})$, where ρ is an arbitrary l -core and ϕ a stability condition ensuring smoothness. We explicitly construct the \mathbb{C}^* -fixed points in these varieties as equivalence classes of certain quiver representations. We show that the fixed points are in a natural bijection with $\mathcal{P}_{\rho^t}(nl + |\rho^t|)$, the set of partitions of $nl + |\rho^t|$ with l -core ρ^t . In particular, the fixed points in $\mathcal{X}_{\theta}(n\delta)$ are naturally labelled by partitions of nl with a trivial l -core. This is the content of section 12.

Having classified the fixed points in all the varieties appearing in (28), we can now consider the induced correspondences between the fixed points step by step. We first determine the bijection

$(\text{Spec } Z_{0,\mathbf{h}})^{\mathbb{C}^*} \rightarrow (\mathcal{X}_\theta(n\delta))^{\mathbb{C}^*}$ induced by the Etingof-Ginzburg isomorphism. We will show that the inverse of the Etingof-Ginzburg map sends a partition of nl with a trivial l -core to the reverse of its l -quotient. The proof of this fact is rather involved and occupies sections 13 and 14.

We then consider the bijection $(\mathcal{X}_\phi(n\delta + \mathbf{d}_\rho))^{\mathbb{C}^*} \rightarrow (\mathcal{X}_{\sigma_i \cdot \phi}(n\delta + \mathbf{d}_{\sigma_i * \rho}))^{\mathbb{C}^*}$ induced by the Nakajima reflection functor \mathfrak{R}_i . In terms of labelling sets we have bijections

$$\mathcal{P}_{\rho^t}(nl + |\rho^t|) \rightarrow \mathcal{P}_{(\sigma_i * \rho)^t}(nl + |(\sigma_i * \rho)^t|). \quad (32)$$

We show that this bijection is given by the affine symmetric group action defined in §10.2. The proof of this fact occupies most of section 15. The only task left is to determine the bijection

$$\left(\mathcal{X}_{-\frac{1}{2}}(\gamma)\right)^{\mathbb{C}^*} \longrightarrow (\text{Hilb}(\nu))^{\mathbb{C}^*}. \quad (33)$$

To do this we will compare the \mathbb{C}^* -characters of fibres of tautological vector bundles on these two spaces. These characters are given by the residues of partitions labelling the fixed points. In particular, distinct fixed points give rise to distinct characters. Using $U(1)$ -equivariance, we conclude that (33) sends a partition to its transpose.

Part III: Correspondence between the \mathbb{C}^* -fixed points

12 \mathbb{C}^* -fixed points in quiver varieties

In this section we explicitly construct the \mathbb{C}^* -fixed points in the quiver varieties $\mathcal{X}_\theta(n\delta + \mathbf{d}_\nu)$, assuming smoothness, as conjugacy classes of quadruples of certain matrices. Our description generalizes the work of Wilson, who classified the \mathbb{C}^* -fixed points in the special case $l = 1$ in [29, Proposition 6.11]. Our construction depends on the Frobenius form of a partition. In §12.1 we define the matrices representing the fixed points in the special case when a partition consists of a single Frobenius hook. In §12.2 we define more general matrices for arbitrary partitions. We then define in §12.3 a basis with respect to which our quadruples of matrices are to be interpreted as quiver representations. We show that isomorphism classes of these quiver representations are in fact fixed under the \mathbb{C}^* -action. We finish by computing the character of the fibre of the tautological bundle at each fixed point.

Notation. We remind the reader that the subscript in θ_i should always be considered modulo l .

12.1 The matrix $A(m, r)$. Suppose that M is a matrix. We let $M[i, j]$ denote the entry of M in the i -th row (counting from the top) and j -th column (counting from the left). We say that $M[i, j]$ lies on the $(j - i)$ -th diagonal.

Definition 12.1. Let $m \geq 1$ and $1 \leq r \leq m$. We let $\Lambda(m)$ denote the $m \times m$ matrix with 1's on the first diagonal and all other entries equal to 0. Let $A(m, r)$ denote the $m \times m$ matrix whose only nonzero entries lie on the (-1) -st diagonal and satisfy

$$A(m, r)[j + 1, j] = \begin{cases} \sum_{i=1}^j \theta_{r-i} & \text{if } 1 \leq j < r \\ -\sum_{i=0}^{m-j-1} \theta_{-m+r+i} & \text{if } r \leq j \leq m-1. \end{cases}$$

Lemma 12.2. *The matrix $[\Lambda(m), A(m, r)]$ is diagonal with eigenvalues*

$$[\Lambda(m), A(m, r)][j, j] = \begin{cases} \theta_{r-j} & \text{if } 1 \leq j \neq r \leq m \\ -\sum_{i=1}^{r-1} \theta_{r-i} - \sum_{i=0}^{m-r-1} \theta_{-m+r+i} & \text{if } j = r. \end{cases}$$

Proof. Let $\alpha_j := A(m, r)[j+1, j]$. Then $\Lambda(m)A(m, r) = \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_{m-1}, 0)$ and $A(m, r)\Lambda(m) = \text{diag}(0, \alpha_1, \alpha_2, \dots, \alpha_{m-1})$. In particular, $[\Lambda(m), A(m, r)] = \text{diag}(\alpha_1, \alpha_2 - \alpha_1, \dots, \alpha_{m-1} - \alpha_{m-2}, -\alpha_{m-1})$. \square

Example 12.3. Let $l = 3, m = 8, r = 5$. Then $A(m, r)$ is the following matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta_1 + \theta_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta_2 + \theta_1 + \theta_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_2 + 2\theta_1 + \theta_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\theta_2 - \theta_1 - \theta_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\theta_1 - \theta_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\theta_0 & 0 \end{pmatrix}.$$

12.2 The matrix $A(\mu)$. Let $\nu \in \heartsuit(l)$ and $\mu \in \mathcal{P}_\nu(nl + |\nu|)$. Let us write it in the Frobenius notation $\mu = (a_1, \dots, a_k \mid b_1, \dots, b_k)$, where k is the number of Frobenius hooks in the Young diagram of μ , and a_i (b_i) is the arm (leg) length of the i -th longest hook. For each $1 \leq i \leq k$, let $r_i = b_i + 1$, $m_i = a_i + b_i + 1$ and $\beta_i = \theta_0 + \sum_{i=1}^{r_i-1} \theta_{r_i-i} + \sum_{i=0}^{m-r_i-1} \theta_{-m+r_i+i}$.

Definition 12.4. We define $A(\mu)$ to be the matrix with diagonal blocks $A(\mu)_{ii} = A(m_i, r_i)$ and off-diagonal blocks $A(\mu)_{ij}$, where $A(\mu)_{ij}$ is the unique $m_i \times m_j$ matrix with nonzero entries only on the $(r_j - r_i - 1)$ -st diagonal satisfying

$$\Lambda(m_i)A(\mu)_{ij} - A(\mu)_{ij}\Lambda(m_j) = -\beta_i E(r_i, r_j), \quad (34)$$

where $E(r_i, r_j)$ is the $m_i \times m_j$ matrix unit with $E(r_i, r_j)[s, t] = 0$ unless $s = r_i, t = r_j$ and $E(r_i, r_j)[r_i, r_j] = 1$.

Explicitly, if $i > j$ then the non-zero diagonal of $A(\mu)_{ij}$ has r_i entries equal to β_i followed by $m_i - r_i$ entries equal to zero. If $i < j$ then the non-zero diagonal of $A(\mu)_{ij}$ has $r_j - 1$ entries equal to 0 followed by $n_j - r_j + 1$ entries equal to $-\beta_i$.

Definition 12.5. Let $\Lambda(\mu) = \bigoplus_{i=1}^k \Lambda(m_i)$. Setting $q_i = \sum_{s=1}^{i-1} m_s + r_i$, let $J(\mu)$ be the $nl \times 1$ matrix with entry β_i in the q_i -th row (for $1 \leq i \leq k$) and all other entries zero. Furthermore, let $I(\mu)$ be the $1 \times nl$ matrix with entry 1 in the q_i -th column (for $1 \leq i \leq k$) and all other entries zero. Finally, we set

$$\mathbf{A}(\mu) := (\Lambda(\mu), A(\mu), I(\mu), J(\mu)).$$

Example 12.6. Let $l = 3$ and $\mu = (3, 1 \mid 2, 1)$. Then $m_1 = 6, m_2 = 3$ and $r_1 = 3, r_2 = 2$. Set $h = \theta_2 + \theta_1 + \theta_0$. Then $A(\mu)$ is the matrix

$$\left(\begin{array}{cccccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \theta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta_2 + \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\theta_2 - \theta_1 - \theta_0 & 0 & 0 & 0 & 0 & -2h & 0 & 0 \\ 0 & 0 & 0 & -\theta_1 - \theta_0 & 0 & 0 & 0 & 0 & -2h & 0 \\ 0 & 0 & 0 & 0 & -\theta_0 & 0 & 0 & 0 & 0 & 0 \\ \hline h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h & 0 & 0 & 0 & 0 & \theta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\theta_2 & 0 & 0 \end{array} \right).$$

12.3 The fixed points. Recall that we have chosen an l -core ν . Let $\mathbf{d}_{\nu^t} = (d_0, \dots, d_{l-1})$ be the dimension vector corresponding to its transpose. Set $\mathbf{d} = n\delta + \mathbf{d}_{\nu^t} = (n + d_0, \dots, n + d_{l-1})$ and $\mathbf{d}' = (1, n + d_0, \dots, n + d_{l-1})$. Let \mathbf{V}_i^ν be a complex vector space of dimension $n + d_i$ for $i = 0, \dots, l-1$. Additionally, let \mathbf{V}_∞ be a complex vector space of dimension one. Set $\widehat{\mathbf{V}}^\nu = \bigoplus_{i=0}^{l-1} \mathbf{V}_i^\nu$ and

$\mathbf{V}^\nu = \widehat{\mathbf{V}}^\nu \oplus \mathbf{V}_\infty$. We regard these as graded vector spaces. Assume that the parameter θ is chosen so that the variety $\mathcal{X}_\theta(n\delta + \mathbf{d}_{\nu^t})$ is smooth.

Our goal now is to interpret $\mathbf{A}(\mu)$ as a quiver representation. With this goal in mind we choose a suitable ordered basis of the vector space \mathbf{V}^ν . We show that the endomorphisms of \mathbf{V}^ν defined by $\mathbf{A}(\mu)$ with regard to this basis respect the quiver grading and thus constitute a quiver representation. We next show that this quiver representation lies in the fibre of the moment map at $\tilde{\theta}$. This allows us to conclude that the conjugacy class of $\mathbf{A}(\mu)$ is a point in the quiver variety $\mathcal{X}_\theta(n\delta + \mathbf{d}_{\nu^t})$. We finish by showing that this point is fixed under the \mathbb{C}^* -action.

Definition 12.7. Consider the sequence $S := (1, \dots, m_1, 1, \dots, m_2, \dots, 1, \dots, m_k)$. We call an (increasing) subsequence of the form $(1, \dots, m_i)$ the i -th *block* of this sequence and denote it by S_i . Let u_j be the j -th element in S . Let $\zeta : \{1, \dots, nl + |\nu|\} \rightarrow \{1, \dots, k\}$ be the function given by the rule

$$\zeta(j) = i \iff u_j \in S_i.$$

If $i, j \in \mathbb{N}$, let $\delta(i, j) = 1$ if $i = j$ and $\delta(i, j) = 0$ otherwise. For each $1 \leq i \leq nl + |\nu|$ let

$$\psi(i) = (r_{\zeta(i)} - u_i) \bmod l.$$

For each $0 \leq j \leq l-1$ and $0 \leq i \leq nl + |\nu|$, let $\tau_j(i)$ be defined recursively according to the formula

$$\tau_j(0) = 0, \quad \tau_j(i) = \tau_j(i-1) + \delta(\psi(i), j).$$

For each $0 \leq i \leq l-1$, let us choose a basis $\{v_i^1, \dots, v_i^{n+d_i}\}$ of \mathbf{V}_i^ν . We now define a function

$$\text{Bas} : \{1, \dots, nl + |\nu|\} \rightarrow \{v_i^{j_i} \mid 0 \leq i \leq l-1, 1 \leq j_i \leq n + d_i\}, \quad i \mapsto v_{\psi(i)}^{\tau_{\psi(i)}(i)}.$$

We also define a function $\text{Cell} : \{1, \dots, nl + |\nu|\} \rightarrow \mathbb{Y}(\mu^t)$ associating to a natural number i a cell in the Young diagram of μ . We define $\text{Cell}(i)$ to be the u_i -th cell in the $\zeta(i)$ -th Frobenius hook of μ^t , counting from the hand of the hook, moving to the left towards the root of the hook and then down towards the foot.

Lemma 12.8. *The functions Cell and Bas are bijections.*

Proof. The fact that Cell is a bijection follows directly from the definitions. Observe that $\psi(i)$ equals the l -content of $\text{Cell}(i)$. We thus have a commutative diagram

$$\begin{array}{ccc} \{1, \dots, nl + |\nu|\} & \xrightarrow{\text{Bas}} & \{v_i^{j_i} \mid 0 \leq i \leq l-1, 1 \leq j_i \leq n + d_i\} \\ \text{Cell} \downarrow & & \downarrow v_i^{j_i} \mapsto i \\ \mathbb{Y}(\mu^t) & \xrightarrow{l\text{-content}} & \{0, \dots, l-1\}. \end{array}$$

By [13, Theorem 2.7.41], the l -residue of μ^t equals $\sum_{i=0}^{l-1} (n + d_i)t^i$ because the l -core of μ^t is ν^t . Hence for each $0 \leq i \leq l-1$ there are exactly $n + d_i$ elements $s \in \{1, \dots, nl + |\nu|\}$ such that the l -content of $\text{Cell}(s)$ equals i . By the commutativity of our diagram, we conclude that there are exactly $n + d_i$ elements $s \in \{1, \dots, nl + |\nu|\}$ such that $\text{Bas}(s) \in \mathbf{V}_i^\nu$.

Now suppose that $s < s'$ and $\text{Bas}(s), \text{Bas}(s') \in \mathbf{V}_i^\nu$. Then $\psi(s) = \psi(s')$. Since $s < s'$ and the function $\tau_{\psi(s')}(-)$ is non-decreasing we have $\tau_{\psi(s')}(s') = \tau_{\psi(s')}(s'-1) + 1 > \tau_{\psi(s')}(s'-1) \geq \tau_{\psi(s)}(s)$. Hence $\text{Bas}(s) \neq \text{Bas}(s')$. We conclude that the function Bas is injective. Since the domain and codomain have the same cardinality, Bas is also bijective. \square

Definition 12.9. Let $\mathbb{B} := (\text{Bas}(1), \text{Bas}(2), \dots, \text{Bas}(nl + |\nu|))$. By Lemma 12.8, \mathbb{B} is an ordered basis of $\widehat{\mathbf{V}}^\nu$. From now on we consider the matrices $\Lambda(\mu)$ and $A(\mu)$ as linear endomorphisms of $\widehat{\mathbf{V}}^\nu$ relative to the ordered basis \mathbb{B} . Let us choose a nonzero vector $v_\infty \in \mathbf{V}_\infty$. We consider the matrix $I(\mu)$ as a linear transformation $\widehat{\mathbf{V}}^\nu \rightarrow \mathbf{V}_\infty$ relative to the ordered bases $\{v_\infty\}$ and \mathbb{B} . We also consider the matrix $J(\mu)$ as a linear transformation $\mathbf{V}_\infty \rightarrow \widehat{\mathbf{V}}^\nu$ relative to the ordered bases \mathbb{B} and $\{v_\infty\}$.

Let $\mu \in \mathcal{P}_\nu(nl + |\nu|)$. Suppose that $\mu = (a_1, \dots, a_k \mid b_1, \dots, b_k)$ is the Frobenius form of μ . As before, set $r_i = b_i + 1$, $m_i = a_i + b_i + 1$ and $q_i = \sum_{j < i} m_j + r_i$.

Lemma 12.10. *Suppose that $1 \leq i \leq k$. Let "LC" stand for "linear combination".*

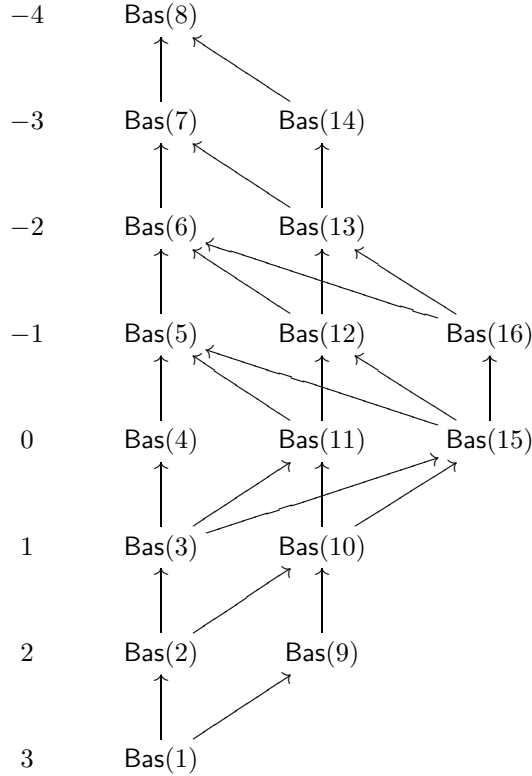
- *If $0 \leq j < a_i$ then $A(\mu)(\text{Bas}(q_i + j))$ is a LC of $\text{Bas}(q_1 + j + 1), \text{Bas}(q_2 + j + 1), \dots, \text{Bas}(q_i + j + i)$,*
- *if $0 \leq j = a_i$ then $A(\mu)(\text{Bas}(q_i + j))$ is a LC of $\text{Bas}(q_1 + j + 1), \text{Bas}(q_2 + j + 1), \dots, \text{Bas}(q_{i-1} + j + i)$,*
- *if $0 > j \geq -b_i$ then $A(\mu)(\text{Bas}(q_i + j))$ is a LC of $\mathbf{1}_{-j \leq b_i + 1} \text{Bas}(q_i + j + 1), \mathbf{1}_{-j \leq b_{i+1} + 1} \text{Bas}(q_{i+1} + j + 1), \dots, \mathbf{1}_{-j \leq b_k + 1} \text{Bas}(q_k + j + i)$.*

Here $\mathbf{1}_{-j \leq b_k + 1}$ is an indicator function taking value one if $-j \leq b_k + 1$ and zero otherwise.

Proof. This is immediate from Definition 12.4. \square

Lemma 12.10 has a very elegant diagrammatic interpretation. We will explain it by means of an example.

Example 12.11. Consider $\mu = (5, 5, 4, 2)$. The Frobenius form of μ is $(4, 3, 1 \mid 3, 2, 0)$. We have $q_1 = 4, q_2 = 11, q_3 = 15$. The diagram below should be interpreted in the following way: $A(\mu)(\text{Bas}(j))$ is a LC of the vectors $\text{Bas}(i)$ such that there is an arrow $\text{Bas}(j) \rightarrow \text{Bas}(i)$.



We have also introduced a numbering of the rows of the diagram. It is easy to see that $\text{Bas}(j) \in \mathbf{V}_i^\nu$ if and only if $\text{Bas}(j)$ lies in a row whose number is congruent to i modulo l .

Lemma 12.12. *Let $\mu \in \mathcal{P}_\nu(nl + |\nu|)$. Then $\mathbf{A}(\mu) \in R(\mathbf{d}')$.*

Proof. We need to check that for each $0 \leq i \leq l - 1$:

$$\text{Im}(A(\mu)|_{\mathbf{V}_i^\nu}) \subseteq \mathbf{V}_{i-1}^\nu, \quad \text{Im}(\Lambda(\mu)|_{\mathbf{V}_i^\nu}) \subseteq \mathbf{V}_{i+1}^\nu,$$

$$\text{Im}(J(\mu)) \subseteq \mathbf{V}_0^\nu, \quad \bigoplus_{i=1}^{l-1} \mathbf{V}_i^\nu \subseteq \ker(I(\mu)).$$

Let us first show that for each $0 \leq i \leq l-1$ we have $\text{Im}(A(\mu)|_{\mathbf{V}_i^\nu}) \subseteq \mathbf{V}_{i-1}^\nu$. We can draw a diagram as in Example 12.11. The subspace \mathbf{V}_i^ν has a basis consisting of vectors $\text{Bas}(j)$ in rows whose number is congruent to i modulo l . The diagram shows that $A(\mu)(\text{Bas}(j))$ is a LC of basis vectors in the row above $\text{Bas}(j)$. But that row has number congruent to $i-1$ modulo l . Hence $A(\mu)(\text{Bas}(j)) \in \mathbf{V}_{i-1}^\nu$.

The argument for $\Lambda(\mu)$ is analogous (but simpler) so we omit it. We now prove the last claim. Let $p \in \{1, \dots, nl + |\nu|\}$ and suppose that $\text{Bas}(p) \notin \mathbf{V}_0^\nu$. Let $1 \leq j \leq k$ and set $q_j = \sum_{s=1}^{j-1} m_s + r_j$. Since $\psi(q_j) = r_j - r_j = 0$ we conclude that $p \notin \{q_1, \dots, q_k\}$. But the only non-zero entries of $I(\mu)$ are those in columns numbered q_j , for $1 \leq j \leq k$. Hence $\text{Bas}(p) \in \ker I(\mu)$. The calculation for $J(\mu)$ is similar. □

Proposition 12.13. *Let $\mu \in \mathcal{P}_\nu(nl + |\nu|)$. Then $\mathbf{A}(\mu) \in \mu_{\mathbf{d}}^{-1}(\tilde{\theta})$.*

Proof. By the previous lemma, we know that $\mathbf{A}(\mu) \in R(\mathbf{d}')$. Lemma 12.2 together with (34) immediately imply that $[\Lambda(\mu), A(\mu)] + J(\mu)I(\mu) = \theta$, so $\mathbf{A}(\mu) \in \mu_{\mathbf{d}}^{-1}(\tilde{\theta})$. □

Theorem 12.14. *Let $\mu \in \mathcal{P}_\nu(nl + |\nu|)$. Then $[\mathbf{A}(\mu)] := G(\mathbf{d}).\mathbf{A}(\mu)$ is a \mathbb{C}^* -fixed point in the quiver variety $\mathcal{X}_\theta(n\delta + \mathbf{d}_{\nu^t})$.*

Proof. Let $t \in \mathbb{C}^*$. We have $t.\mathbf{A}(\mu) = (t^{-1}\Lambda(\mu), tA(\mu), I(\mu), J(\mu))$. We need to find a matrix M in $G(\mathbf{d})$ such that $Mt.\mathbf{A}(\mu)M^{-1} = \mathbf{A}(\mu)$.

For every $t \in \mathbb{C}^*$, let $Q(t) = \text{diag}(1, t^{-1}, \dots, t^{-nl+1})$. Conjugating an $nl \times nl$ matrix by $Q(t)$ multiplies the j -th diagonal by t^j . In particular, we have $Q(t)(\bigoplus_{i=1}^k tA(m_i, r_i))Q(t)^{-1} = (\bigoplus_{i=1}^k A(m_i, r_i))$ and $Q(t)t^{-1}\Lambda(\mu)Q(t)^{-1} = \Lambda(\mu)$.

Now consider the effect of conjugating $A(\mu)$ by $Q(t)$ on the off-diagonal block $A(\mu)_{ij}$ ($i \neq j$). This block contains only one nonzero diagonal. Counting within the block, it is the diagonal number $r_j - r_i - 1$. Counting inside the entire matrix $A(\mu)$, it is the diagonal number $q_j - q_i - 1$, where $q_i = m_1 + \dots + m_{i-1} + r_i$. It follows that conjugation by $Q(t)$ multiplies the block $A(\mu)_{ij}$ by $t^{q_j - q_i - 1}$. Hence we have

$$Q(t) \left(\bigoplus_{1 \leq i \neq j \leq k} tA(\mu)_{ij} \right) Q(t)^{-1} = \bigoplus_{1 \leq i \neq j \leq k} t^{q_j - q_i} A(\mu)_{ij}.$$

Let $P(t) = \bigoplus_{i=1}^k t^{q_i} \text{id}_{m_i}$. Conjugating $A(\mu)$ by $P(t)$ doesn't change the diagonal blocks but multiplies each off-diagonal block $A(\mu)_{ij}$ by $t^{q_i - q_j}$. We conclude that

$$P(t)Q(t)tA(\mu)Q(t)^{-1}P(t)^{-1} = A(\mu).$$

Since the matrix $\Lambda(\mu)$ contains only diagonal blocks, conjugating by $P(t)$ doesn't have any impact. Hence

$$P(t)Q(t)t^{-1}\Lambda(\mu)Q(t)^{-1}P(t)^{-1} = \Lambda(\mu).$$

The nonzero rows of $J(\mu)$ are precisely rows number q_1, q_2, \dots, q_k . But the q_i -th entry of $P(t)$ is t^{q_i} and the q_i -th entry of $Q(t)$ is t^{1-q_i} . Hence $P(t)Q(t)J(\mu) = tJ(\mu)$. Similarly, $I(\mu)Q(t)^{-1}P(t)^{-1} = t^{-1}I(\mu)$. Let $D(t) = t^{-1}\text{id}_{nl}$. Since $D(t)$ is a scalar matrix, conjugating by $D(t)$ doesn't change $A(\mu)$ or $\Lambda(\mu)$. On the other hand, $D(t)P(t)Q(t)J(\mu) = J(\mu)$ and $I(\mu)Q(t)^{-1}P(t)^{-1}D(t)^{-1} = I(\mu)$.

The matrices $D(t), Q(t), P(t)$ are diagonal, so they represent linear automorphisms in $G(\mathbf{d})$. Hence $\mathbf{A}(\mu)$ and $t.\mathbf{A}(\mu)$ lie in the same $G(\mathbf{d})$ -orbit, which is equivalent to saying that $\mathbf{A}(\mu)$ is a \mathbb{C}^* -fixed point in $\mathcal{N}_{\mathbf{h}}$. □

12.4 Characters of the fibres of \mathcal{V} at the fixed points. Let \mathcal{V}_μ denote the fibre of the tautological bundle \mathcal{V} at the fixed point $[\mathbf{A}(\mu)] = G(\mathbf{d}).\mathbf{A}(\mu)$.

Proposition 12.15. *Let $\mu \in \mathcal{P}_\nu(nl + |\nu|)$. Then*

$$\text{ch}_t \mathcal{V}_\mu = \sum_{\square \in \mu} t^{c(\square)} = \text{Res}_\mu(t).$$

Proof. Let $[(\mathbf{A}(\mu), v)] \in \mu_{\mathbf{d}}^{-1}(\tilde{\theta}) \times^{G(\mathbf{d})} \hat{\mathbf{V}}^\nu = \mathcal{V}$. We have

$$\begin{aligned} t.(\mathbf{A}(\mu), v) &= (t.\mathbf{A}(\mu), v) \sim (D(t)P(t)Q(t)(t.\mathbf{A}(\mu))Q(t)^{-1}P(t)^{-1}D(t)^{-1}, Q(t)^{-1}P(t)^{-1}D(t)^{-1}v) \\ &= (\mathbf{A}(\mu), Q(t)^{-1}P(t)^{-1}D(t)^{-1}v). \end{aligned}$$

The basis vectors $\{\text{Bas}(1), \text{Bas}(2), \dots, \text{Bas}(nl + |\nu|)\}$ are eigenvectors of $(D(t)P(t)Q(t))^{-1}$ with corresponding eigenvalues

$$\{t^{1-r_1}, t^{2-r_1}, \dots, t^{m_1-r_1}, t^{1-r_2}, t^{2-r_2}, \dots, t^{m_2-r_2}, \dots, t^{1-r_k}, t^{2-r_k}, \dots, t^{m_k-r_k}\}. \quad (35)$$

Note that these are just the diagonal entries of the matrix $(D(t)P(t)Q(t))^{-1}$. However, these numbers are precisely the contents of the cells in the Young diagram of μ , counting from the foot of the innermost Frobenius hook upward and later to the right, before passing to subsequent Frobenius hooks. Hence $\text{ch}_t \mathcal{V}_\mu = \sum_{\square \in \mu} t^{c(\square)} = \text{Res}_\mu(t)$. \square

Recall that we have made the assumption that the parameter θ is chosen so that the variety $\mathcal{X}_\theta(n\delta + \mathbf{d}_{\nu^t})$ is smooth. Suppose there exists $w \in \tilde{S}_l$ such that $w * \mathbf{d}_{\nu^t} = 0$. Let $w = \sigma_{i_1} \dots \sigma_{i_m}$ be a reduced expression for w in \tilde{S} . Furthermore, let $w \cdot \theta = (\vartheta_0, \dots, \vartheta_{l-1})$ and $H_1 = \vartheta_1, \dots, H_{l-1} = \vartheta_{l-1}$, $h = -\sum_{i=0}^{l-1} \vartheta_i$, $\mathbf{h} = (h, H_1, \dots, H_{l-1})$. Composing the Etingof-Ginzburg map with reflection functors we obtain a \mathbb{C}^* -equivariant isomorphism

$$\text{Spec } Z_{0, \mathbf{h}} \xrightarrow{\text{EG}} \mathcal{N}_{\mathbf{h}} = \mathcal{X}_{w \cdot \theta}(n\delta) \xrightarrow{\mathfrak{R}_{i_m} \circ \dots \circ \mathfrak{R}_{i_1}} \mathcal{X}_\theta(n\delta + \mathbf{d}_{\nu^t}).$$

Corollary 12.16. *Let $\mathcal{X}_\theta(n\delta + \mathbf{d}_{\nu^t})^{\mathbb{C}^*}$ denote the set of closed \mathbb{C}^* -fixed points in $\mathcal{X}_\theta(n\delta + \mathbf{d}_{\nu^t})$. The map*

$$\mathcal{P}_\nu(nl + |\nu|) \rightarrow \mathcal{X}_\theta(n\delta + \mathbf{d}_{\nu^t})^{\mathbb{C}^*}, \quad \mu \mapsto [\mathbf{A}(\mu)] = G(\mathbf{d}).\mathbf{A}(\mu) \quad (36)$$

is a bijection.

Proof. The \mathbb{C}^* -fixed points in $\text{MaxSpec } Z_{0, \mathbf{h}}$ are in bijection with l -multipartitions of n , which are themselves in bijection with partitions of $nl + |\nu|$ with l -core ν . But $\text{Spec } Z_{0, \mathbf{h}}$ is \mathbb{C}^* -equivariantly isomorphic to $\mathcal{X}_\theta(n\delta + \mathbf{d}_{\nu^t})$, so $|\mathcal{X}_\theta(n\delta + \mathbf{d}_{\nu^t})^{\mathbb{C}^*}| = |(\text{MaxSpec } Z_{0, \mathbf{h}})^{\mathbb{C}^*}| = |\mathcal{P}(l, n)| = |\mathcal{P}_\nu(nl + |\nu|)|$.

Since a partition is uniquely determined by its residue, $\mu \neq \mu'$ implies $\text{ch}_t \mathcal{V}_\mu \neq \text{ch}_t \mathcal{V}_{\mu'}$, by Proposition 12.15, which in turn implies that $[\mathbf{A}(\mu)] \neq [\mathbf{A}(\mu')]$. It follows that (36) is a bijection because it is an injective function between sets of the same cardinality. \square

13 Degenerate affine Hecke algebras

We have obtained an explicit classification of the \mathbb{C}^* -fixed points in $\text{Spec } Z_{0, \mathbf{h}}$ and the Calogero-Moser space $\mathcal{N}_{\mathbf{h}} = \mathcal{X}_\theta(n\delta)$. Our next goal is to describe the correspondence between them under the Etingof-Ginzburg map. In this section we use degenerate affine Hecke algebras and a version of the Chevalley restriction map to associate to each fixed point a multiset in \mathbb{C}^n/S_n in a manner which is compatible with the Etingof-Ginzburg isomorphism. More precisely, to each multipartition $\underline{\Delta}$ we associate a multiset $\rho_1(\underline{\Delta})$ and to each partition μ we associate a multiset $\rho_2(\mu)$. We can conclude that $\text{EG}(\underline{\Delta}) = \mu$ if $\rho_1(\underline{\Delta}) = \rho_2(\mu)$.

We begin by recalling some results about degenerate affine Hecke algebras, denoted \mathcal{H}_κ , and their representation theory. The embedding $\mathcal{H}_\kappa \hookrightarrow \mathbb{H}_{0, \mathbf{h}}$ allows us to construct the map ρ_1 mentioned above. We use the Chevalley restriction theorem to obtain ρ_2 . We then use the restriction functor $\mathbb{H}_{0, \mathbf{h}}\text{-mod} \rightarrow \mathcal{H}_\kappa\text{-mod}$ to show that $\rho_1 = \rho_2 \circ \text{EG}$. We conclude this section by calculating the images of the \mathbb{C}^* -fixed points under ρ_1 and ρ_2 .

13.1 Degenerate affine Hecke algebras. Degenerate affine Hecke algebras (dAHA's) associated to generalized symmetric groups were constructed in [25]. We now recall their definition and basic properties.

Definition 13.1. Let $\kappa \in \mathbb{C}$. The degenerate affine Hecke algebra associated to Γ_n is defined to be the \mathbb{C} -algebra $\mathcal{H}_\kappa := \mathcal{H}_\kappa(\Gamma_n)$ generated by Γ_n and pairwise commuting elements z_1, \dots, z_n satisfying the following relations:

$$\begin{aligned} \epsilon_j z_i &= z_i \epsilon_j \quad (1 \leq i, j \leq n), \quad s_{i,i+1} z_j = z_j s_{i,i+1} \quad (j \neq i, i+1), \\ s_{i,i+1} z_{i+1} &= z_i s_{i,i+1} + \kappa \sum_{k=0}^{l-1} \epsilon_i^{-k} \epsilon_{i+1}^k \quad (1 \leq i \leq n-1). \end{aligned}$$

We let \mathcal{Z}_κ denote the centre of \mathcal{H}_κ .

Proposition 13.2. *The algebra \mathcal{H}_κ has the following properties.*

- (i) *As a vector space, \mathcal{H}_κ is canonically isomorphic to $\mathbb{C}[z_1, \dots, z_n] \otimes \mathbb{C}\Gamma_n$. We call this isomorphism the PBW isomorphism.*
- (ii) *There is an injective algebra homomorphism $\mathbb{C}[z_1, \dots, z_n]^{S_n} \hookrightarrow \mathcal{Z}_\kappa$.*
- (iii) *The algebra \mathcal{H}_κ has a maximal commutative subalgebra \mathcal{C}_κ which is isomorphic to $\mathbb{C}[z_1, \dots, z_n] \otimes \mathbb{C}(\mathbb{Z}/l\mathbb{Z})^n$.*
- (iv) *Suppose that $h = \kappa$. Then there exists an injective algebra homomorphism $\mathcal{H}_\kappa \hookrightarrow \mathbb{H}_{0,h}$ defined by*

$$g \mapsto g \quad (g \in \Gamma_n), \quad z_i \mapsto x_i y_i + \kappa \sum_{1 \leq j < i} \sum_{k=0}^{l-1} s_{i,j} \epsilon_i^k \epsilon_j^{-k} + \sum_{k=1}^{l-1} c_k \sum_{m=0}^{l-1} \eta^{-mk} \epsilon_i^m,$$

where the c_k 's are the parameters obtained from \mathbf{h} as in (5). This homomorphism restricts to a homomorphism $\mathbb{C}[z_1, \dots, z_n]^{S_n} \hookrightarrow Z_{0,h}$.

Proof. See [5, Proposition 2.1, Proposition 2.3, Section 3.1, Proposition 1.1] and [12, Proposition 10.1, Corollary 10.1]. \square

In particular, the homomorphism $\mathcal{H}_\kappa \hookrightarrow \mathbb{H}_{0,h}$ sends

$$z_1 \mapsto x_1 y_1 + \sum_{k=1}^{l-1} c_k \sum_{m=0}^{l-1} \eta^{-mk} \epsilon_1^m.$$

Definition 13.3. Let $\rho_1 : \text{Spec } Z_{0,h} \rightarrow \mathbb{C}^n/S_n$ be the dominant morphism induced by the embedding $\mathbb{C}[z_1, \dots, z_n]^{S_n} \hookrightarrow Z_{0,h}$.

13.2 Simple modules over dAHA's. We now recall the construction of principal series modules over \mathcal{H}_κ and the criterion for their simplicity.

Definition 13.4. Let $a = (a_1, \dots, a_n) \in \mathbb{C}^n$ and $b = (b_1, \dots, b_n) \in (\mathbb{Z}/l\mathbb{Z})^n$. Let $\mathbb{C}_{a,b}$ be the one-dimensional representation of the commutative algebra $\mathcal{C}_\kappa = \mathbb{C}[z_1, \dots, z_n] \otimes \mathbb{C}(\mathbb{Z}/l\mathbb{Z})^n$ defined by

$$z_i \cdot v = a_i v, \quad \epsilon_i \cdot v = \eta^{b_i} v$$

for each $1 \leq i \leq n$ and $v \in \mathbb{C}_{a,b}$. We define the *principal series module* associated to the parameters a, b to be the induced \mathcal{H}_κ -module

$$M(a, b) := \mathcal{H}_\kappa \otimes_{\mathcal{C}_\kappa} \mathbb{C}_{a,b}.$$

Proposition 13.5. *Let $a \in \mathbb{C}^n$ and $b \in (\mathbb{Z}/l\mathbb{Z})^n$. If $a_i - a_j \neq 0, \pm l\kappa$ for all $1 \leq i \neq j \leq n$ then the \mathcal{H}_κ -module $M(a, b)$ is irreducible.*

Proof. See [5, Theorem 4.9]. \square

13.3 Restricting $\mathbb{H}_{0,\mathbf{h}}$ -modules to \mathcal{H}_h -modules. Let us now set $\kappa = h$. We are going to consider the generic behaviour of simple modules over $\mathbb{H}_{0,\mathbf{h}}$ under the restriction functor to \mathcal{H}_h -modules.

Definition 13.6. Set

$$\mathcal{D} = \{a = (a_1, \dots, a_n) \in \mathbb{C}^n \mid a_i - a_j \neq 0, \pm l\kappa \text{ for all } 1 \leq i \neq j \leq n\}.$$

Observe that \mathcal{D} is a dense open subset of \mathbb{C}^n . Proposition 13.5 implies that for all $a \in \mathcal{D}$ and $b \in (\mathbb{Z}/l\mathbb{Z})^n$ the module $M(a, b)$ is irreducible.

Definition 13.7. Consider the diagram

$$\mathbb{C}^n \xrightarrow{\phi} \mathbb{C}^n/S_n \xleftarrow{\rho_1} \text{Spec } Z_{0,\mathbf{h}}$$

We define $\mathcal{U} := \rho_1^{-1}(\phi(\mathcal{D}))$.

Lemma 13.8. *The subset \mathcal{U} is open and dense in $\text{Spec } Z_{0,\mathbf{h}}$.*

Proof. The fact that \mathcal{U} is open follows immediately from the fact that ϕ is a quotient map and ρ_1 is continuous.

Since the morphism ρ_1 is dominant, $\rho_1(\text{Spec } Z_{0,\mathbf{h}})$ is dense in \mathbb{C}^n/S_n . Since $\phi(\mathcal{D})$ is open in \mathbb{C}^n/S_n we have $\phi(\mathcal{D}) \cap \rho_1(\text{Spec } Z_{0,\mathbf{h}}) \neq \emptyset$. Hence \mathcal{U} is nonempty. The fact that the variety $\text{Spec } Z_{0,\mathbf{h}}$ is irreducible (Proposition 3.5) now implies that \mathcal{U} is dense. \square

Let $\hat{e} = \frac{1}{(n-1)!} \sum_{g \in S_{n-1} \subset \Gamma_n} g$ and $\mathbf{0} = (0, \dots, 0) \in (\mathbb{Z}/l\mathbb{Z})^n$.

Lemma 13.9. *For each irreducible $\mathbb{H}_{0,\mathbf{h}}$ -module L whose support is contained in \mathcal{U} (i.e. $\chi_L \in \mathcal{U}$), there exists an injective homomorphism of \mathcal{H}_h -modules*

$$M(a, \mathbf{0}) \hookrightarrow L$$

for some $a \in \mathcal{D}$.

Proof. Suppose that L is a simple $\mathbb{H}_{0,\mathbf{h}}$ -module whose support is in \mathcal{U} . Using the embedding $\mathcal{H}_h \hookrightarrow \mathbb{H}_{0,\mathbf{h}}$ we consider L as a module over \mathcal{H}_h . We have a $(\mathbb{Z}/l\mathbb{Z})^n$ -module decomposition $L = \bigoplus_{b \in (\mathbb{Z}/l\mathbb{Z})^n} L(b)$, where $b = (b_1, \dots, b_n)$ and $L(b)$ is the subspace of L such that $\epsilon_i \cdot w = \eta^{b_i} w$ for all $w \in L(b)$. Since the z_i 's commute with the ϵ_j 's, each subspace $L(b)$ is preserved under the action of the z_i 's. In particular, z_1, \dots, z_n define commuting linear operators on $L(\mathbf{0})$, so they have some common eigenvector $v \in L(\mathbf{0})$. Let a_1, \dots, a_n be the respective eigenvalues of the z_i 's. Since the support of L is contained in \mathcal{U} , we have $a = (a_1, \dots, a_n) \in \mathcal{D}$.

Let $v_{a,\mathbf{0}} \in \mathbb{C}_{a,\mathbf{0}}$. Then the map $1 \otimes v_{a,\mathbf{0}} \mapsto v$ defines a \mathcal{H}_h -module homomorphism $M(a, \mathbf{0}) \rightarrow L$. Since $a = (a_1, \dots, a_n) \in \mathcal{D}$, the module $M(a, \mathbf{0})$ is simple and so this homomorphism is injective. \square

Lemma 13.10. *Suppose that L is an irreducible $\mathbb{H}_{0,\mathbf{h}}$ -module whose support is contained in \mathcal{U} so that there exists an injective \mathcal{H}_h -module homomorphism $M(a, \mathbf{0}) \hookrightarrow L$ for some $a = (a_1, \dots, a_n) \in \mathcal{D}$. Then $\hat{e}M(a, \mathbf{0}) \subset L^{\Gamma_{n-1}}$. Moreover, $\hat{e}M(a, \mathbf{0})$ is stable under the action of z_1 and the eigenvalues of z_1 on $\hat{e}M(a, \mathbf{0})$ are a_1, \dots, a_n .*

Proof. We have a vector space isomorphism $M(a, \mathbf{0}) \cong \mathbb{C}S_n \otimes \mathbb{C}_{a,\mathbf{0}}$. Therefore $\{\hat{e}s_{1,j} \otimes v_{a,\mathbf{0}} \mid 1 \leq j \leq n\}$ form a basis of $\hat{e}M(a, \mathbf{0})$ for some $v_{a,\mathbf{0}} \in \mathbb{C}_{a,\mathbf{0}}$. We now show that each of these basis elements is fixed under the action of Γ_{n-1} . We first note that since for each $g \in S_{n-1} \subset \Gamma_n$ we have $g\hat{e} = \hat{e}$, the subgroup S_{n-1} fixes each $\hat{e}s_{1,j} \otimes v_{a,\mathbf{0}}$. Now consider $\epsilon_i \cdot \hat{e}s_{1,j} \otimes v_{a,\mathbf{0}}$ with $2 \leq i \leq n$. We have $\epsilon_i \cdot \hat{e}s_{1,j} \otimes v_{a,\mathbf{0}} = \sum_{g \in S_{n-1}} g s_{1,j} \epsilon_{i(g)} \otimes v_{a,\mathbf{0}}$, where $i(g)$ is an index depending on g . But each $\epsilon_{i(g)}$ acts on $v_{a,\mathbf{0}}$ by the identity, so we conclude that ϵ_i fixes $\hat{e}s_{1,j} \otimes v_{a,\mathbf{0}}$. The stability of $\hat{e}M(a, \mathbf{0})$ under the action of z_1 follows from the fact that z_1 commutes with \hat{e} .

Using the relations in Definition 13.1 and induction one can show that

$$z_1 s_1 s_2 \dots s_{j-1} = s_1 s_2 \dots s_{j-1} z_j - \sum_{i=1}^{j-1} s_1 \dots \gamma_i \hat{s}_i \dots s_{j-1},$$

where the hat denotes omission and $\gamma_i = \kappa \sum_{k=0}^{l-1} \epsilon_i^{-k} \epsilon_{i+1}^k$. Recall that

$$s_{1,j} = s_1 \dots s_{j-2} s_{j-1} s_{j-2} \dots s_1.$$

Since z_j commutes with s_{j-2}, \dots, s_1 we get

$$z_1 s_{1,j} = s_{1,j} z_j - \sum_{i=1}^{j-1} s_1 \dots \gamma_i \hat{s}_i \dots s_{j-1} s_{j-2} \dots s_1.$$

Using the commutation relations in \mathcal{H}_h we can write

$$s_1 \dots \gamma_i \hat{s}_i \dots s_{j-1} s_{j-2} \dots s_1 = s_1 \dots \hat{s}_i \dots s_{j-1} s_{j-2} \dots s_1 \tilde{\gamma}_i$$

with $\tilde{\gamma}_i \in \mathbb{C}(\mathbb{Z}/l\mathbb{Z})^n$. But $(\mathbb{Z}/l\mathbb{Z})^n$ acts trivially on the vector $v_{a,\mathbf{0}}$ so we can ignore all the γ_i terms. For each $1 \leq i < j-1$ we have

$$s_1 \dots \hat{s}_i \dots s_{j-1} s_{j-2} \dots s_1 = (1, j, i+1)$$

and $s_1 \dots \hat{s}_{j-1} s_{j-2} \dots s_1 = 1$. Hence $z_1 s_{1,i} = s_{1,i} z_i - 1 - \sum_{i=1}^{j-2} (1, j, i+1)$. Noting that $\hat{e}(1, j, i+1) = \hat{e} s_{1,i+1}$ we conclude that $z_1 \hat{e} s_{1,j} = \hat{e} z_1 s_{1,j} = \hat{e} s_{1,j} z_j - \sum_{i=1}^{j-1} \hat{e} s_{1,i}$ and so

$$z_1 \hat{e} s_{1,j} \otimes v_{a,\mathbf{0}} = a_j \hat{e} s_{1,j} \otimes v_{a,\mathbf{0}} - \sum_{i=1}^{j-1} \hat{e} s_{1,i} \otimes v_{a,\mathbf{0}}.$$

It follows that the action of z_1 with respect to the basis $\{\hat{e} s_{1,j} \otimes v_{a,\mathbf{0}} \mid 1 \leq j \leq n\}$ is given by the upper-triangular matrix

$$\begin{pmatrix} a_1 & -1 & \dots & -1 \\ 0 & a_2 & & \dots \\ \dots & & \dots & -1 \\ 0 & \dots & 0 & a_n \end{pmatrix}.$$

□

13.4 A commutative diagram. Suppose that $\hat{e}M(a, \mathbf{0})$ is as in Lemma 13.10. Since ϵ_1 acts trivially on $\hat{e}M(a, \mathbf{0})$, we can identify $\hat{e}M(a, \mathbf{0})$ with \mathbf{V}_0 using isomorphisms $L^{\Gamma_{n-1}} \cong (\mathbb{C}\Gamma_n)^{\Gamma_{n-1}}$ and (14). The action of the operator z_1 on $\hat{e}M(a, \mathbf{0})$ can therefore be identified, under the Etingof-Ginzburg isomorphism, with the matrix $Y_1 X_0$ (up to conjugation), where $\text{EG}(\ker \chi_L) = [(\mathbf{X}, \mathbf{Y}, I, J)]$ and $\mathbf{X} = (X_0, \dots, X_{l-1})$, $\mathbf{Y} = (Y_0, \dots, Y_{l-1})$.

Definition 13.11. Let $\rho_2 : \mathcal{N}_h \rightarrow \mathbb{C}^n/S_n$ be the morphism sending $(\mathbf{X}, \mathbf{Y}, I, J)$ to the multiset of the generalized eigenvalues of the matrix $Y_1 X_0$.

Recall the diagram (31) associating to a point in a quiver variety a real number. The following diagram, which will play a crucial role in our argument, can be regarded as an enhancement of (31). We attach a *multiset* of complex numbers rather than just a single number to every point of $\text{Spec } Z_{0,h}$ and \mathcal{N}_h .

Lemma 13.12. *The diagram*

$$\begin{array}{ccc} \text{Spec } Z_{0,h} & \xrightarrow{\text{EG}} & \mathcal{N}_h \\ & \searrow \rho_1 \quad \swarrow \rho_2 & \\ & \mathbb{C}^n/S_n & \end{array}$$

commutes.

Proof. Since \mathbf{EG} is an isomorphism, it suffices to show there exists a dense open subset of $\mathrm{Spec} Z_{0,\mathbf{h}}$ on which the diagram is commutative. Consider the dense open subset \mathcal{U} from Definition 13.7. For each $\chi \in \mathcal{U}$ we have $\chi = \chi_L$ for a unique simple module L and there is an injective \mathcal{H}_h -module homomorphism $M(a, \mathbf{0}) \hookrightarrow L$ for some $a \in \mathcal{D}$, by Lemma 13.9. Set $\mathbf{EG}(\ker \chi_L) =: [(\mathbf{X}, \mathbf{Y}, I, J)]$ with $\mathbf{X} = (X_0, \dots, X_{l-1})$, $\mathbf{Y} = (Y_0, \dots, Y_{l-1})$. The remarks at the beginning of this section imply that the matrix $Y_1 X_0$ describes the action of z_1 on $\hat{e}M(a, \mathbf{0})$. Hence the eigenvalues of $Y_1 X_0$ are the same as the eigenvalues of the operator $z_1|_{\hat{e}M(a, \mathbf{0})}$. By Lemma 13.10 these eigenvalues are a_1, \dots, a_n . Hence $\rho_2 \circ \mathbf{EG}(\ker \chi_L) = a \in \mathbb{C}^n/S_n$.

On the other hand, consider the composition

$$\mathbb{C}[z_1, \dots, z_n]^{S_n} \hookrightarrow Z_{0,\mathbf{h}} \xrightarrow{\chi_L} \mathbb{C}. \quad (37)$$

By the definition of $M(a, \mathbf{0})$, a symmetric polynomial $f(z_1, \dots, z_n)$ acts on $1 \otimes \mathbb{C}_{a,\mathbf{0}}$ by the scalar $f(a_1, \dots, a_n)$. Since $f(z_1, \dots, z_n)$ is central in $\mathbb{H}_{0,\mathbf{h}}$, it acts by this scalar on all of L . Therefore, the kernel of (37) equals the maximal ideal in $\mathbb{C}[z_1, \dots, z_n]^{S_n}$ consisting of those symmetric polynomials f which satisfy $f(a_1, \dots, a_n) = 0$. But this ideal corresponds to the point $a = (a_1, \dots, a_n)$ in $\mathrm{Spec} \mathbb{C}[z_1, \dots, z_n]^{S_n} = \mathbb{C}^n/S_n$. \square

13.5 The images of the fixed points in \mathbb{C}^n/S_n . We are now going to identify the images of the \mathbb{C}^* -fixed points under the maps ρ_1 and ρ_2 . Let

$$s_0 = 0, \quad s'_0 = -h, \quad s_i = s'_i = \sum_{j=1}^i H_j \quad (i = 1, \dots, l-1),$$

$$s''_i = h + \sum_{j=1}^{l-i-1} H_j \quad (i = 0, \dots, l-2), \quad s''_{l-1} = 0.$$

$$\mathbf{s} = (s_0, s_1, \dots, s_{l-1}), \quad \mathbf{s}' = (s'_0, s'_1, \dots, s'_{l-1}), \quad \mathbf{s}'' = (s''_0, s''_1, \dots, s''_{l-1}).$$

Also recall that $\theta_0 = -h + H_0$ and $\theta_1 = H_1, \dots, \theta_{l-1} = H_{l-1}$.

Notation. Let $(a_1, \dots, a_n) \in \mathbb{C}^n/S_n$. We identify this multiset with the Laurent polynomial $\sum_{i=1}^n t^{a_i}$.

Lemma 13.13. *Let $\underline{\Delta} \in \mathcal{P}(l, n)$. Then $\rho_1(\mathrm{Ann}(\underline{\Delta})) = \mathrm{Res}_{\underline{\Delta}}^{\mathbf{s}}(t^h)$.*

Proof. See [19, §5.4]. \square

Definition 13.14. Let $\mu \in \mathcal{P}_{\mathcal{O}}(nl)$ and let $\mu = (a_1, \dots, a_k \mid b_1, \dots, b_k)$ be its Frobenius form. If $\Lambda_{m_i} A(m_i, r_i) = \mathrm{diag}(\alpha_1, \alpha_2, \dots, \alpha_{m_i-1}, \alpha_{m_i})$, then we define

$$\mathrm{Eig}(\mu, i) = \sum_{\substack{1 \leq j \leq m_i, \\ j \equiv r_i - 1 \pmod{l}}} t^{\alpha_j}, \quad \mathrm{Eig}(\mu) = \sum_{i=1}^k \mathrm{Eig}(\mu, i).$$

Lemma 13.15. *Let $\mu \in \mathcal{P}_{\mathcal{O}}(nl)$ and let $\mu = (a_1, \dots, a_k \mid b_1, \dots, b_k)$ be its Frobenius form. We have*

$$\rho_2([\mathbf{A}(\mu)]) = \mathrm{Eig}(\mu) = \sum_{i=1}^k \left(\left(t^{s'_{b_i \bmod l}} \sum_{j=1}^{\lceil b_i/l \rceil} t^{-(j-1)h} \right) + \left(t^{s''_{a_i \bmod l}} \sum_{j=1}^{\lfloor (a_i+1)/l \rfloor} t^{(j-1)h} \right) \right). \quad (38)$$

Proof. $\mathrm{Eig}(\mu)$ picks out exactly the eigenvalues of the restricted endomorphism $\Lambda(\mu)A(\mu)|_{\mathbf{v}_1}$ from all the eigenvalues of $\Lambda(\mu)A(\mu)$. But these are the same as the eigenvalues of $A(\mu)\Lambda(\mu)|_{\mathbf{v}_0}$. The fact that $\rho_2([\mathbf{A}(\mu)]) = \mathrm{Eig}(\mu)$ now follows immediately from the definition of the morphism ρ_2 .

For the second equality it suffices to show that for each $i = 1, \dots, k$ we have

$$\mathrm{Eig}(\mu, i) = \left(t^{s'_{b_i \bmod l}} \sum_{j=1}^{\lceil b_i/l \rceil} t^{-(j-1)h} \right) + \left(t^{s''_{a_i \bmod l}} \sum_{j=1}^{\lfloor (a_i+1)/l \rfloor} t^{(j-1)h} \right). \quad (39)$$

We can write

$$\text{Eig}(\mu, i) = \sum_{\substack{1 \leq j \leq m_i, \\ j=r_i-1 \bmod l}} t^{\alpha_j} = \sum_{\substack{1 \leq j \leq r_i-1, \\ j=r_i-1 \bmod l}} t^{\alpha_j} + \sum_{\substack{r_i \leq j \leq m_i, \\ j=r_i-1 \bmod l}} t^{\alpha_j}.$$

Note that $r_i - 1 = b_i = l \cdot (\lceil b_i/l \rceil - 1) + d_i$, where d_i is an integer such that $1 \leq d_i \leq l$. The j 's satisfying $1 \leq j \leq r_i - 1$ and $j = r_i - 1 \bmod l$ are therefore precisely $b_i, b_i - l, b_i - 2l, \dots, b_i - (\lceil b_i/l \rceil - 1) \cdot l = d_i$. Recall that $\theta_0 + \theta_1 + \dots + \theta_{l-1} = -h$. Hence $\alpha_{d_i+pl} = \alpha_{d_i} - ph$ for $p = 0, \dots, \lceil b_i/l \rceil - 1$, by Definition 12.1. Therefore

$$\sum_{\substack{1 \leq j \leq r_i-1, \\ j=r_i-1 \bmod l}} t^{\alpha_j} = t^{\alpha_{d_i}} \sum_{j=1}^{\lceil b_i/l \rceil} t^{-(j-1)h}.$$

Now observe that $d_i = b_i \bmod l$ if $1 \leq d_i < l$. Hence $\alpha_{d_i} = \alpha_{b_i \bmod l} = \sum_{j=1}^{b_i \bmod l} \theta_{b_i+1-j \bmod l} = \sum_{j=1}^{b_i \bmod l} \theta_j = s'_{b_i \bmod l}$. If $d_i = l$ then $\alpha_{d_i} = \alpha_l = \sum_{j=1}^l \theta_{b_i+1-j \bmod l} = \sum_{j=1}^l \theta_j = -h = s'_0$. This shows that

$$\sum_{\substack{1 \leq j \leq r_i-1, \\ j=r_i-1 \bmod l}} t^{\alpha_j} = t^{s'_{b_i \bmod l}} \sum_{j=1}^{\lceil b_i/l \rceil} t^{-(j-1)h}.$$

We have $m_i - r_i + 1 = a_i + 1 = l \cdot \lfloor (a_i + 1)/l \rfloor + e_i$ with $0 \leq e_i < l$. The j 's satisfying $r_i \leq j \leq m_i$ and $j = r_i - 1 \bmod l$ are therefore precisely $b_i + l, b_i + 2l, \dots, b_i + \lfloor (a_i + 1)/l \rfloor \cdot l$. Note that $b_i + \lfloor (a_i + 1)/l \rfloor \cdot l = m_i - e_i$. Hence $\alpha_{m_i-e_i-pl} = \alpha_{m_i-e_i} + ph$ for $p = 0, \dots, \lfloor (a_i + 1)/l \rfloor - 1$. One computes, in a similar fashion as above, that $\alpha_{m_i-e_i} = s''_{a_i \bmod l}$. This shows that

$$\sum_{\substack{r_i \leq j \leq m_i, \\ j=r_i-1 \bmod l}} t^{\alpha_j} = t^{s''_{a_i \bmod l}} \sum_{j=1}^{\lfloor (a_i+1)/l \rfloor} t^{(j-1)h}.$$

□

Let $\mu \in \mathcal{P}_{\emptyset}(nl)$. By Lemma 13.12, Lemma 13.15, Lemma 13.13 and the fact that a multipartition is uniquely determined by its \mathbf{s} -residue for generic \mathbf{s} , we have

$$\text{Eig}(\mu) = \rho_2([\mathbf{A}(\mu)]) = \rho_1(\text{Ann}(\underline{\lambda})) = \text{Res}_{\underline{\lambda}}^{\mathbf{s}}(t^h) \quad (40)$$

for a unique $\underline{\lambda} \in \mathcal{P}(l, n)$.

Definition 13.16. We define $\underline{\text{Eig}}(\mu) = (\underline{\text{Eig}}(\mu)^0, \underline{\text{Eig}}(\mu)^1, \dots, \underline{\text{Eig}}(\mu)^{l-1}) \in \mathcal{P}(l, n)$ by the equation

$$\text{Res}_{\underline{\text{Eig}}(\mu)}^{\mathbf{s}}(t^h) = \text{Eig}(\mu).$$

We thus have a bijection

$$\underline{\text{Eig}} : \mathcal{P}_{\emptyset}(nl) \rightarrow \mathcal{P}(l, n), \quad \mu \mapsto \underline{\text{Eig}}(\mu).$$

Corollary 13.17. Let $\mu \in \mathcal{P}_{\emptyset}(nl)$. The inverse of the Etingof-Ginzburg isomorphism sends the \mathbb{C}^* -fixed point $[\mathbf{A}(\mu)]$ in $\mathcal{N}_{\mathbf{h}}$ to the \mathbb{C}^* -fixed point $\text{Ann}(\underline{\text{Eig}}(\mu))$ in $\text{Spec } Z_{0, \mathbf{h}}$.

Proof. This follows directly from (40). □

14 Combinatorial induction

In this section we identify the multipartition $\underline{\text{Eig}}(\mu)$ and thereby establish the correspondence between the \mathbb{C}^* -fixed points under the Etingof-Ginzburg map. We also deduce the cyclotomic q -hook formula.

14.1 The strategy. Our next goal is to show that $\text{Eig}(\mu) = (\text{Quot}(\mu))^b$. We will use the following strategy. Recall Lemma 9.7. We first prove that an analogous statement holds for the multipartition $\text{Eig}(\mu)$. This will allow us to argue by induction on n . We then prove that $\text{Eig}(\mu) = (\text{Quot}(\mu))^b$ for partitions μ with the special property that only a unique l -rim-hook can be removed from μ . We then deduce the result for arbitrary $\mu \in \mathcal{P}_\emptyset(nl)$. At the end of this section we collect all the results proven so far and deduce the cyclotomic q -hook formula.

14.2 Types and contributions of Frobenius hooks. We need to introduce some notation to break down the formula (38) into simpler pieces.

Definition 14.1. Let $\mu \in \mathcal{P}_\emptyset(nl)$ and let $\mu = (a_1, \dots, a_k \mid b_1, \dots, b_k)$ be its Frobenius form. Let F_1, \dots, F_k be the Frobenius hooks in $\mathbb{Y}(\mu)$ so that (i, i) is the root of F_i . Let

$$\text{type}_\mu(L, i) := b_i \bmod l, \quad \text{type}_\mu(A, i) := -(a_i + 1) \bmod l \quad (i = 1, \dots, k).$$

We call the number $\text{type}_\mu(L, i)$ the *type* of $\text{leg}(F_i)$ and the number $\text{type}_\mu(A, i)$ the *type* of $\text{arm}(F_i)$. Define

$$\Xi_\mu(L, i) := t^{s'_{b_i \bmod l}} \sum_{j=1}^{\lceil b_i/l \rceil} t^{-(j-1)h}, \quad \Xi_\mu(A, i) := t^{s''_{a_i \bmod l}} \sum_{j=1}^{\lfloor (a_i+1)/l \rfloor} t^{(j-1)h}.$$

We call $\Xi_\mu(L, i)$ the *contribution* of $\text{leg}(F_i)$ and $\Xi_\mu(A, i)$ the *contribution* of $\text{arm}(F_i)$.

By (39) we have

$$\text{Eig}(\mu, i) = \Xi_\mu(L, i) + \Xi_\mu(A, i). \quad (41)$$

Lemma 14.2. Consider the l -multipartition $\text{Eig}(\mu) = (\text{Eig}(\mu)^0, \text{Eig}(\mu)^1, \dots, \text{Eig}(\mu)^{l-1})$. We have

$$t^{s_j} \text{Res}_{\text{Eig}(\mu)^j}(t^h) = \sum_{\substack{1 \leq i \leq k, \\ \text{type}_\mu(A, i) = j}} \Xi_\mu(A, i) + \sum_{\substack{1 \leq i \leq k, \\ \text{type}_\mu(L, i) = j}} \Xi_\mu(L, i). \quad (42)$$

Proof. Each summand t^d on the RHS of (38) corresponds (non-canonically) to a cell in the multipartition $\text{Eig}(\mu)$. Without loss of generality we can assume that all the s_i 's are pairwise distinct, $h \in \mathbb{Z}$ and $0 < s_1, \dots, s_{l-1} < 1$. Then we can write $t^d = t^{s_i} t^e$ for some unique $i = 0, \dots, l-1$ and $e \in \mathbb{Z}$. The summand t^d corresponds to a cell in the partition $\text{Eig}(\mu)^j$ if and only if $i = j$, i.e., $t^d = t^{s_j} t^e$. Since $\text{Eig}(\mu) = \sum_{p=1}^k \text{Eig}(\mu, p)$, formula (41) implies that there exists an $1 \leq p \leq k$ such that t^d is a summand in $\Xi_\mu(L, p)$ or $\Xi_\mu(A, p)$. In the former case $t^d = t^{s_j} t^e$ if and only if $j = b_p \bmod l = \text{type}_\mu(L, p)$. In the latter case $t^d = t^{s_j} t^e$ if and only if $s'_{a_p \bmod l} = h + s_j$, which is the case if and only if $j = -(a_i + 1) \bmod l = \text{type}_\mu(A, i)$. \square

14.3 Removal of rim-hooks. We will now investigate the effect of removing a rim-hook from μ on the multipartition $\text{Eig}(\mu)$. Let $\mathbb{Y}_{+/-/0}(\mu)$ denote the subset of $\mathbb{Y}(\mu)$ consisting of cells of positive/negative/zero content.

Lemma 14.3. Let R be an l -rim-hook in $\mathbb{Y}(\mu)$ and suppose that $R \subset \mathbb{Y}_0(\mu) \cup \mathbb{Y}_+(\mu)$. Suppose that R intersects r Frobenius hooks, labelled F_{p+1}, \dots, F_{r+p} so that (i, i) is the root of F_i . Let $\mu' := \mu - R$. Then

$$\text{type}_{\mu'}(A, j) = \text{type}_\mu(A, j+1), \quad \Xi_{\mu'}(A, j) = \Xi_\mu(A, j+1) \quad (j = p+1, \dots, p+r-1),$$

$$\text{type}_{\mu'}(A, p+r) = \text{type}_\mu(A, p+1), \quad \Xi_{\mu'}(A, p+r) = \Xi_\mu(A, p+1) - M,$$

where M is the (monic) monomial in $\Xi_\mu(A, p)$ of highest degree in h .

Proof. It is clear that R must intersect adjacent Frobenius hooks. Recall that the residue of R is of the form $\text{Res}_R(t) = \sum_{i=i_0}^{i=i_0+l-1} t^i$ with $i_0 \geq 0$. Moreover, we have

$$\text{Res}_{R \cap F_j}(t) = \sum_{i=i_{r+p-j}}^{i_{r+p-j+1}-1} t^i \quad (43)$$

for some integers $i_0 < i_1 < \dots < i_r = i_0 + l$. One can easily see that these integers satisfy

$$i_{r+p-j+1} - 1 = a_j, \quad (44)$$

where $a_j = |\text{arm}(F_j)| = \max_{\square \in F_j} c(\square)$. Set $d_j := \max_{\square \in F_j - R} c(\square)$. If $F_j - R = \emptyset$ set $d_j = -1$. From (43) and (44) we easily deduce that

$$d_j = a_{j+1} \quad (j = p+1, \dots, p+r-1), \quad d_{p+r} = a_{p+1} - l. \quad (45)$$

By definition, the type and contribution of $\text{arm}(F_j)$ resp. $\text{arm}(F_j - R)$ depend only on the numbers a_j and d_j . The lemma now follows immediately from the definitions. \square

The reader may consult Figure 1 for a visual interpretation of Lemma 14.3.

Proposition 14.4. *Let R be a rim-hook in μ and set $\mu' := \mu - R$. Then $\underline{\text{Eig}}(\mu') = \underline{\text{Eig}}(\mu) - \blacksquare$ for some $\blacksquare \in \underline{\text{Eig}}(\mu)$.*

Proof. There are three possibilities: $R \subset \mathbb{Y}_0(\mu) \cup \mathbb{Y}_+(\mu)$, $R \subset \mathbb{Y}_0(\mu) \cup \mathbb{Y}_-(\mu)$ or $R \cap \mathbb{Y}_+(\mu) \neq \emptyset, R \cap \mathbb{Y}_-(\mu) \neq \emptyset$.

Consider the first case. Lemma 14.3 and Lemma 14.2 imply that there exists a $j \in \{0, \dots, l-1\}$ such that $\underline{\text{Eig}}(\mu')^i = \underline{\text{Eig}}(\mu)^i$ if $i \neq j$ and $t^{s_j} \text{Res}_{\underline{\text{Eig}}(\mu')^j}(t^h) = t^{s_j} \text{Res}_{\underline{\text{Eig}}(\mu)^j}(t^h) - t^{s_j} M$ for some monic monomial $M = t^{qh} \in \mathbb{Z}[t^h]$. Hence $\underline{\text{Eig}}(\mu') = \underline{\text{Eig}}(\mu) - \blacksquare$ for some $\blacksquare \in \underline{\text{Eig}}(\mu)$ with $c(\blacksquare) = q$.

The second case is analogous. Now consider the third case. We claim that $\Xi(A, i) = 0$ for every Frobenius hook F_i whose arm intersects R nontrivially. Indeed, by definition $\Xi(A, i) \neq 0$ only if $|\text{arm}(F_i)| + 1 \geq l$. We have $|\text{arm}(F_i)| = \max_{\square \in \text{arm}(F_i)} c(\square) = \max_{\square \in \text{arm}(F_i) \cap R} c(\square)$. Hence $|\text{arm}(F_i)| \leq \max_{\square \in R} c(\square)$. However, since $R \cap \mathbb{Y}_-(\mu) \neq \emptyset$, the rim-hook R must contain a cell of content -1 . The fact that $\text{Res}_R(t) = t^q \sum_{p=0}^{l-1} t^p$ for some $q \in \mathbb{Z}$ implies that $\max_{\square \in R} c(\square) \leq l-2$. Hence $|\text{arm}(F_i)| + 1 \leq l-1$ and so $\Xi(A, i) = 0$. Therefore the removal of R does not affect the contribution of the arm of any Frobenius hook.

Now set $R' := R \cap \mathbb{Y}_0(\mu) \cup \mathbb{Y}_-(\mu)$. We have reduced the third case back to the second case, with the modification that R' is now a truncated rim-hook. We can still apply Lemma 14.3 with minor adjustments. In particular, equations (44) are still true with the exception that the final equation becomes $d_{p+r} = 0$. Let j be the smallest integer such that $\text{leg}(F_j) \cap R \neq \emptyset$. Using the same argument as before, we conclude that $t^{s_j} \text{Res}_{\underline{\text{Eig}}(\mu')^j}(t^h) = t^{s_j} \text{Res}_{\underline{\text{Eig}}(\mu)^j}(t^h) - t^{s_j-h}$ \square

14.4 Partitions with a unique removable rim-hook. In this section we will show that $\underline{\text{Eig}}(\mu) = (\text{Quot}(\mu))^b$ for a certain class of partitions which we call *l-special*.

Definition 14.5. We say that a partition μ is *l-special* if the rim of $\mathbb{Y}(\mu)$ contains a unique *l*-rim-hook R . We call R the *unique removable l-rim-hook* in $\mathbb{Y}(\mu)$. Let $\mathcal{P}_{\emptyset}^{sp}(k)$ denote the set of partitions of k which are *l-special* and have a trivial *l*-core. More generally, we say that a cell $\square \in \mathbb{Y}(\mu)$ is *removable* if $\mathbb{Y}(\mu) - \square$ is the Young diagram of a partition.

Our goal now is to describe partitions of nl which are *l-special* and have a trivial *l*-core. Throughout this section we assume that $\mu \in \mathcal{P}_{\emptyset}^{sp}(nl)$. We let R denote the unique removable *l*-rim-hook in $\mathbb{Y}(\mu)$ and set $\mu' := \mu - R$.

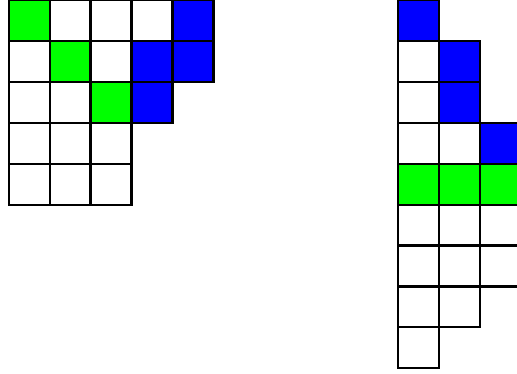


Figure 1: The figure on the left shows the Young diagram of the partition $(5, 5, 4, 3, 3)$. The cells of content zero are marked with green. The blue cells form a 4-rim-hook. The figure on the right shows the same Young diagram rearranged so that cells of the same content occupy the same row. Using this visual representation we can easily determine the impact of removing the blue rim-hook. The length of the first arm after the removal equals the length of the second arm before the removal. Similarly, the length of the second arm after the removal equals the length of the third arm before the removal. Finally, the length of the third arm after the removal equals the length of the first arm before the removal minus four. This is precisely the content of Lemma 14.3.

Lemma 14.6. (i) Every column of $\mathbb{B}(\mu)$ contains the same number of beads.
(ii) Sliding distinct beads up results in the removal of distinct l -rim-hooks from $\mathbb{Y}(\mu)$.
(iii) The bead diagram $\mathbb{B}(\mu)$ contains $l - 1$ columns with no gaps and one column with a unique string of adjacent gaps.

Proof. (i) The empty bead diagram describes the trivial partition. But bead diagrams which describe the trivial partition and have the property that the number of beads in the diagram is divisible by l are unique up to adding or deleting full rows at the top of the diagram. Hence any such diagram consists of consecutive full rows at the top. Since μ has a trivial l -core, the process of sliding beads upward in $\mathbb{B}(\mu)$ must result in a bead diagram of this shape. But this is only possible if every column of $\mathbb{B}(\mu)$ contains the same number of beads.

(ii) We can see this by considering the quotients of partitions corresponding to the bead diagrams obtained by moving up distinct beads. If the beads moved are on distinct runners, then a box is removed from distinct partitions in $\text{Quot}(\mu)$, so distinct multipartitions arise. If the beads are on the same runner, sliding upward distinct beads implies changing the first-column hook lengths in different ways in the same partition, so different multipartitions arise as well. But a trivial-core partition is uniquely determined by its quotient, so these distinct multipartitions are quotients of distinct partitions of $l(n - 1)$ obtained by the removal of distinct rim-hooks from $\mathbb{Y}(\mu)$.

(iii) Since only one rim-hook can be removed from μ , only one bead in our bead diagram can be moved upward. This implies that $l - 1$ runners contain no gaps (i.e. they contain a consecutive string of beads counting from the top). The remaining runner must contain a unique gap or a unique string of gaps. \square

Lemma 14.7. The bead diagram $\mathbb{B}(\mu)$ can be decomposed into three blocks A , B and C , counting from the top. Each block consists of identical rows. Rows in block A are full except for one bead. Let's say that the gap due to the absent bead is on runner k . Rows in block B are either all full or all empty. Rows in block C are empty except for one bead on runner k . Moreover, the number of rows in block A equals the number of rows in block C .

Proof. This is an immediate consequence of Lemma 14.6. \square

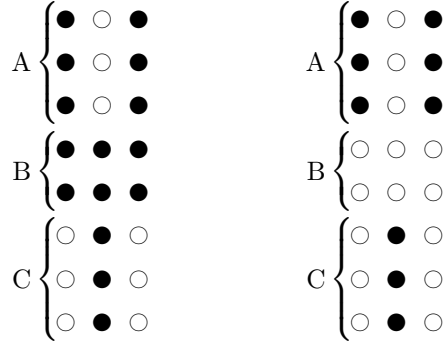


Figure 2: Examples of bead diagrams corresponding to special partitions.

In the sequel we will only consider the case where all the rows in block B are full. All the following claims can easily be adapted to the case of empty rows.

Lemma 14.8. *The Young diagram of the partition μ can be decomposed into four blocks $\hat{A}, \hat{B}, \hat{C}, \hat{D}$. Block \hat{A} is the (Young diagram of the) partition corresponding to the bead diagram A . Suppose that the bead diagram A has m rows and that block B of $\mathbb{B}(\mu)$ has p rows. Then block \hat{B} is a rectangle consisting of m columns and $l \cdot p$ rows. If $k \neq 0$ block \hat{C} is the partition corresponding to the bead diagram C . If $k = 0$ block \hat{C} is the partition corresponding to the bead diagram obtained from C by inserting an extra row at the top, which is full except for the empty point in column $l - 1$. Block \hat{D} is a square with m rows and columns. We recover μ from these blocks by placing \hat{A} at the bottom, stacking \hat{B} on top, then stacking \hat{D} on top and finally placing \hat{C} on the right of \hat{D} .*

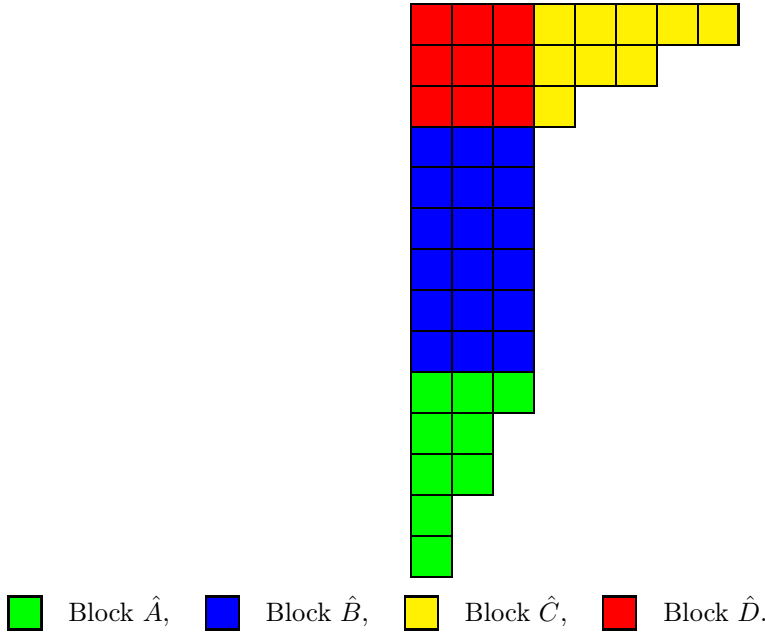


Figure 3: The Young diagram of the partition corresponding to the first bead diagram in Figure 2.

Proof. This follows from Lemma 14.7 by a routine calculation. One easily recovers the first column hook lengths from the positions of the beads. \square

The reader may consult Figure 3 for an example of the decomposition from Lemma 14.8. We

are now ready to investigate the effect of removing the rim-hook R .

Lemma 14.9. *The partition μ can be decomposed into m Frobenius hooks. The unique removable rim-hook lies on the outermost Frobenius hook.*

Proof. By Lemma 14.8, the 0-th diagonal of the Young diagram of μ is contained in \hat{D} and contains m boxes. Hence there are m Frobenius hooks. It follows easily from Lemma 14.7 and Lemma 14.8 that the outermost Frobenius hook in μ is the partition of the form $(k+1, 1^{l \cdot p + l - k - 1})$. In particular, it contains $l \cdot (p+1) \geq l$ cells. Hence it contains a rim-hook. But a rim-hook of the outermost Frobenius hook is a rim-hook of μ (because the outermost Frobenius hook is part of the rim). \square

Proposition 14.10. *We have $\underline{\text{Eig}}(\mu') = \underline{\text{Eig}}(\mu) - \blacksquare$, where \blacksquare is the unique removable cell in the k -th partition in $\underline{\text{Eig}}(\mu)$ with content $-p$.*

Proof. The outermost Frobenius hook F_m in μ is the partition of the form $(k+1, 1^{l \cdot p + l - k - 1})$. By removing the rim-hook R we obtain the partition $(k+1, 1^{l \cdot p - k - 1})$ or the trivial partition if $p = 0$. Since the rim-hook R is contained in the outermost Frobenius hook, its removal does not affect the type and contribution of the arms and legs of the other Frobenius hooks.

There are several cases to be considered. Let $a_m = |\text{arm}(F_m)|$ and $b_m = |\text{leg}(F_m)|$. If $k \neq l-1$, then $\Xi_\mu(L, m) = t^{s'_{b_m \bmod l}} \sum_{i=0}^p t^{-ih}$ and $\text{type}_\mu(L, m) = b_m = l - k - 1$ (while $\Xi_\mu(A, m) = 0$). If $k = l-1$ and $p > 0$ then $\Xi_\mu(L, m) = \sum_{i=1}^p t^{-ih}$ and $\text{type}_\mu(L, m) = 0$ (while $\Xi_\mu(A, m) = 1$). If $k = l-1, p = 0$ then $\Xi_\mu(A, m) = 1$ and $\text{type}_\mu(A, m) = 0$ (while $\Xi_\mu(L, m) = 0$).

Upon removing the rim-hook the polynomials listed above change as follows. We have $\Xi_{\mu'}(L, m) = t^{s'_{b_m \bmod l}} \sum_{i=0}^{p-1} t^{-ih}$ in the first case, $\Xi_{\mu'}(L, m) = \sum_{i=1}^{p-1} t^{-ih}$ in the second case and $\Xi_{\mu'}(A, m) = 0$ in third case. The types do not change. We observe that in each case a monomial of degree t^{-ph} (up to a shift) is subtracted, which corresponds to removing a cell of content $-p$ in $\underline{\text{Eig}}(\mu)^{l-k-1}$. \square

We now obtain an analogous result for the multipartition $\underline{\text{Quot}}(\mu)$.

Lemma 14.11. *$\underline{\text{Quot}}(\mu) = (Q^0(\mu), \dots, Q^{l-1}(\mu))$ is a multipartition consisting of $l-1$ trivial partitions and one non-trivial partition. Suppose that the k -th column in $\mathbb{B}(\mu)$ is the unique column which contains gaps. Then $Q^k(\mu)$ is the unique non-trivial partition in $\underline{\text{Quot}}(\mu)$. If that column has a string of m gaps followed by a string of $q = m + p$ beads then the Young diagram of the $Q^k(\mu)$ is a rectangle consisting of m columns and q rows.*

Proof. The l -quotient of μ can be deduced directly from the bead diagram $\mathbb{B}(\mu)$. The description of the latter in Lemma 14.7 immediately implies the present lemma. \square

Recall that $\mu' := \mu - R$, where R is the unique rim-hook which can be removed from μ .

Lemma 14.12. *We have $\underline{\text{Quot}}(\mu') = \underline{\text{Quot}}(\mu) - \square$, where \square is the box in the bottom right corner of the rectangle described in Lemma 14.11. That box has content $m - q = m - p + m = -p$.*

Proof. This is the only cell which can be removed from $\underline{\text{Quot}}(\mu)$, so the claim follows by Lemma 9.7. \square

The lemma implies in particular that $(\underline{\text{Quot}}(\mu'))^b = (\underline{\text{Quot}}(\mu))^b - \square$, where \square is a box of content $-p$ in the $(l-k-1)$ -th partition in $(\underline{\text{Quot}}(\mu))^b$.

Proposition 14.13. *Suppose that $(\underline{\text{Quot}}(\lambda))^b = \underline{\text{Eig}}(\lambda)$ for any partition $\lambda \vdash l(n-1)$ with trivial l -core. Let $\mu \in \mathcal{P}_{\mathcal{O}}^{sp}(\mu)$. Then $(\underline{\text{Quot}}(\mu))^b = \underline{\text{Eig}}(\mu)$.*

Proof. By induction, $(\underline{\text{Quot}}(\mu'))^b = \underline{\text{Eig}}(\mu')$. But by Proposition 14.10 and Lemma 14.12 both $(\underline{\text{Quot}}(\mu))^b$ and $\underline{\text{Eig}}(\mu)$ arise from $(\underline{\text{Quot}}(\mu'))^b = \underline{\text{Eig}}(\mu')$ by adding a box of content $-p$ to the $(l-k-1)$ -th partition. Hence $(\underline{\text{Quot}}(\mu))^b = \underline{\text{Eig}}(\mu)$. \square

Remark 14.14. Note that the inductive hypothesis in the preceding proposition has not yet been proven. The reason for this is that by removing a rim-hook from a special partition we may obtain a partition from which several distinct rim-hooks can be removed. This case is treated below. Nonetheless, the proposition above yields the base for the overall induction. Indeed, if $n = 1$ then μ is itself a rim-hook.

14.5 Induction. We now generalize Proposition 14.13 to arbitrary partitions.

Theorem 14.15. *Suppose that $(\text{Quot}(\lambda))^b = \text{Eig}(\lambda)$ for any partition $\lambda \vdash l(n-1)$ with trivial l -core. Suppose that we can remove two distinct (but possibly overlapping) rim-hooks R' and R'' from the partition μ . Then $(\text{Quot}(\mu))^b = \text{Eig}(\mu)$.*

Proof. Let $\mu' = \mu - R'$ and $\mu'' = \mu - R''$. Then $\mu' \neq \mu''$ and so $\text{Quot}(\mu') \neq \text{Quot}(\mu'')$ (because the quotient of a partition with trivial core determines that partition uniquely). By Lemma 9.7 we have

$$(\text{Quot}(\mu'))^b = (\text{Quot}(\mu))^b - \square, \quad (\text{Quot}(\mu''))^b = (\text{Quot}(\mu))^b - \hat{\square}$$

with $\square \neq \hat{\square} \in (\text{Quot}(\mu))^b$. By Proposition 14.4 we have

$$\text{Eig}(\mu') = \text{Eig}(\mu) - \blacksquare, \quad \text{Eig}(\mu'') = \text{Eig}(\mu) - \hat{\blacksquare}$$

for some $\blacksquare, \hat{\blacksquare} \in \text{Eig}(\mu)$. We know that Eig establishes a bijection between l -partitions of $n-1$ and partitions of $l(n-1)$ with a trivial l -core. Hence $\blacksquare \neq \hat{\blacksquare}$. By the inductive hypothesis in our lemma,

$$(\text{Quot}(\mu'))^b = \text{Eig}(\mu'), \quad (\text{Quot}(\mu''))^b = \text{Eig}(\mu'').$$

Hence

$$(\text{Quot}(\mu))^b - \square = \text{Eig}(\mu) - \blacksquare, \quad (\text{Quot}(\mu))^b - \hat{\square} = \text{Eig}(\mu) - \hat{\blacksquare}$$

and so

$$\text{Eig}(\mu) = (\text{Quot}(\mu))^b - \square + \blacksquare = (\text{Quot}(\mu))^b - \hat{\square} + \hat{\blacksquare}.$$

Since $\square \neq \hat{\square}$ and $\blacksquare \neq \hat{\blacksquare}$ we conclude that $\square = \blacksquare$ and $\hat{\square} = \hat{\blacksquare}$. Therefore $\text{Eig}(\mu) = (\text{Quot}(\mu))^b$. \square

Corollary 14.16. *Let $\mu \in \mathcal{P}_{\emptyset}(nl)$. Then $(\text{Quot}(\mu))^b = \text{Eig}(\mu)$.*

Proof. This follows immediately from Proposition 14.13 and Theorem 14.15. \square

Corollary 14.17. *The Etingof-Ginzburg map induces a bijection between the labelling sets of the \mathbb{C}^* -fixed points given by*

$$\mathcal{P}(l, n) \rightarrow \mathcal{P}(nl), \quad (\text{Quot}(\mu))^b \mapsto \mu.$$

Proof. By Corollary 13.17, the Etingof-Ginzburg isomorphism sends the \mathbb{C}^* -fixed point $\text{Ann}(\text{Eig}(\mu))$ to $[\mathbf{A}(\mu)]$. According to Corollary 14.16, $(\text{Quot}(\mu))^b = \text{Eig}(\mu)$. \square

14.6 The cyclotomic q -hook formula. We now bring together all the results we have proven so far to deduce the cyclotomic q -hook formula.

Theorem 14.18. *Let $\mu \in \mathcal{P}_{\emptyset}(nl)$ be a partition of nl with a trivial l -core. We have the equality*

$$\sum_{\square \in \mu} t^{c(\square)} = [nl]_t \sum_{\Delta \uparrow (\text{Quot}(\mu))^b} \frac{f_{\Delta}(t)}{f_{(\text{Quot}(\mu))^b}(t)}.$$

Proof. Since the Etingof-Ginzburg map is \mathbb{C}^* -equivariant, we have

$$\text{ch}_t \mathcal{V}_{\mu} = \text{ch}_t(\mathcal{R}^{\Gamma_{n-1}})_{\underline{\gamma}}, \quad (46)$$

where $\underline{\gamma}$ satisfies $\text{EG}(\text{Ann}(\underline{\gamma})) = [\mathbf{A}(\mu)]$. Proposition 12.15 yields the formula for the LHS of (46) and Theorem 7.10 the formula for the RHS of (46). Corollary 14.17 shows that $\underline{\gamma} = (\text{Quot}(\mu))^b$. \square

14.7 Residue of the quotient. As a corollary, we also obtain the following formula for the residue of the l -quotient of a partition of nl with a trivial l -core.

Corollary 14.19. *Let $\mu = (a_1, \dots, a_k \mid b_1, \dots, b_k) \in \mathcal{P}_\emptyset(nl)$. Let us write $\underline{\text{Quot}}(\mu) = (Q^0, \dots, Q^{l-1})$. We have*

$$\text{Res}_{Q^{l-j-1}}(t) = \sum_{\substack{1 \leq i \leq k, \\ -(a_i+1) \equiv j \pmod l}} t^{p_{a_i} \bmod l} \sum_{m=1}^{\lfloor (a_i+1)/l \rfloor} t^{(m-1)} + \sum_{\substack{1 \leq i \leq k, \\ b_i \equiv j \pmod l}} t^{p'_{b_i} \bmod l} \sum_{m=1}^{\lfloor b_i/l \rfloor} t^{-(m-1)},$$

where $p_i = 1$ for $i = 0, \dots, l-2$ and $p_{l-1} = 0$ while $p'_0 = -1$ and $p'_i = 0$ for $i = 1, \dots, l-1$.

Proof. The RHS of the formula above equals $\text{Res}_{\underline{\text{Eig}}(\mu)^j}(t)$ by (42). But $\underline{\text{Eig}}(\mu)^j = Q^{l-j-1}$ by Corollary 14.16. \square

15 Combinatorics of reflection functors

Assume from now on that $l > 1$. In this section we will finish identifying the correspondence between the \mathbb{C}^* -fixed points under (28). We have already tackled the correspondence induced by the Etingof-Ginzburg map. Now we will consider reflection functors. We continue to use the notation introduced in §10.

15.1 Reflection functors and the fixed points. Let $\nu \in \heartsuit(l)$ and let $\mathbf{d}_\nu = (d_{\nu,0}, \dots, d_{\nu,l-1})$ be the corresponding dimension vector (see §10.2). Assume that $\theta \in \mathbb{Q}^l$ is chosen so that $\theta_i \neq 0$ and the quiver variety $\mathcal{X}_\theta(n\delta + \mathbf{d}_\nu)$ is smooth. The reflection functor

$$\mathfrak{R}_i : \mathcal{X}_\theta(n\delta + \mathbf{d}_\nu) \rightarrow \mathcal{X}_{\sigma_i \cdot \theta}(n\delta + \mathbf{d}_{\sigma_i * \nu})$$

induces a bijection $\mathcal{X}_\theta(n\delta + \mathbf{d}_\nu)^{\mathbb{C}^*} \longleftrightarrow \mathcal{X}_{\sigma_i \cdot \theta}(n\delta + \mathbf{d}_{\sigma_i * \nu})^{\mathbb{C}^*}$ between the \mathbb{C}^* -fixed points. Composing with the bijections from Corollary 12.16, we obtain a bijection

$$\mathbf{R}_i : \mathcal{P}_{\nu^t}(nl + |\nu^t|) \rightarrow \mathcal{P}_{(\sigma_i * \nu)^t}(nl + |(\sigma_i * \nu)^t|). \quad (47)$$

We are going to show that

$$\mathbf{R}_i(\mu) = (\mathbf{T}_i(\mu^t))^t,$$

where $\mathbf{T}_i(\mu^t)$ is the partition obtained from μ^t by adding all i -addable and removing all i -removable cells relative to μ^t , as in Definition 10.2.

15.2 The strategy. Our first goal is to describe the action of reflection functors on the fixed points explicitly in terms of linear algebra. In §15.3, given a partition $\mu \in \mathcal{P}_{\nu^t}(nl + |\nu^t|)$, we endow the vector space $\widehat{\mathbf{V}}^\nu$ with a \mathbb{Z} -grading which we call the " μ -grading". A \mathbb{C}^* -fixed point is characterized uniquely by this grading. In §15.5 we compute the $\mathbf{R}_i(\mu)$ -grading on the vector space $\widehat{\mathbf{V}}^{\sigma_i * \nu}$. In §15.6 and §15.7 we use this calculation to give a combinatorial description of the partition $\mathbf{R}_i(\mu)$.

15.3 The μ -grading. Fix an l -core $\nu \in \heartsuit(l)$ and let $\mathbf{d}_\nu = (d_{\nu,0}, \dots, d_{\nu,l-1})$ be the corresponding dimension vector. Set $\widehat{\mathbf{V}}^\nu := \bigoplus_{i=0}^{l-1} \mathbf{V}_i^\nu$, where $\dim_{\mathbb{C}} \mathbf{V}_i^\nu = n + d_{\nu,i}$. The $\mathbb{Z}/l\mathbb{Z}$ -graded complex vector space $\widehat{\mathbf{V}}^\nu$ is the underlying vector space for representations of the double \overline{Q} of the cyclic quiver with dimension vector $n\delta + \mathbf{d}_\nu$. Furthermore, set $\mathbf{V}^\nu = \widehat{\mathbf{V}}^\nu \oplus \mathbf{V}_\infty$, where $\dim_{\mathbb{C}} \mathbf{V}_\infty = 1$. We are now going to introduce a \mathbb{Z} -grading on $\widehat{\mathbf{V}}^\nu$ which "lifts" the $\mathbb{Z}/l\mathbb{Z}$ -grading.

Definition 15.1. Let $\mu \in \mathcal{P}_{\nu^t}(nl + |\nu^t|)$. We call a \mathbb{Z} -grading $\widehat{\mathbf{V}}^\nu = \bigoplus_{i \in \mathbb{Z}} \mathbf{W}_i$ a μ -grading if it satisfies the following condition:

(C) for each $i \in \mathbb{Z}$ we have

$$A(\mu)(\mathbf{W}_i) \subseteq \mathbf{W}_{i-1}, \quad \Lambda(\mu)(\mathbf{W}_i) \subseteq \mathbf{W}_{i+1}, \quad J(\mu)(\mathbf{V}_\infty) \subseteq \mathbf{W}_0, \quad I(\mu)(\mathbf{W}_0) = \mathbf{V}_\infty.$$

Proposition 15.2. *Let $\mu \in \mathcal{P}_{\nu^t}(nl + |\nu^t|)$. A μ -grading on $\widehat{\mathbf{V}}^\nu$ exists and is unique.*

Proof. We first prove existence. Let $\mu = (a_1, \dots, a_k \mid b_1, \dots, b_k)$ be the Frobenius form of μ . Set $r_i = b_i + 1$, $m_i = a_i + b_i + 1$ and $q_i = \sum_{j < i} m_j + r_i$ for $1 \leq i \leq k$. Recall the ordered basis $\{\text{Bas}(i) \mid 1 \leq i \leq nl + |\nu|\}$ of $\widehat{\mathbf{V}}^\nu$ from §12.3. We now define \mathbf{W}_j by the rule that for each $1 \leq i \leq k$:

$$\text{Bas}(q_i - j) \in \mathbf{W}_j \quad (0 \leq j \leq b_i), \quad \text{Bas}(q_i + j) \in \mathbf{W}_{-j} \quad (1 \leq j \leq a_i). \quad (48)$$

It follows directly from the construction of the matrices $A(\mu)$, $\Lambda(\mu)$, $J(\mu)$ and $I(\mu)$ that this grading satisfies the condition (C) in Definition 15.1. Hence it is a μ -grading. Finally we observe that the definition of $\Lambda(\mu)$ and Lemma 12.10 imply that

$$\mathbf{W}_j = (A(\mu))^j(\mathbf{W}_0) \quad (j < 0), \quad \mathbf{W}_j = (\Lambda(\mu))^j(\mathbf{W}_0) \quad (j > 0). \quad (49)$$

We now prove uniqueness. Let $\widehat{\mathbf{V}}^\nu = \bigoplus_{i \in \mathbb{Z}} \mathbf{U}_i$ be a μ -grading. There exists a vector $v_\infty \in \mathbf{V}_\infty$ such that $\text{Bas}(q_1) + \dots + \text{Bas}(q_k) = J(\mu)(v_\infty) \in \mathbf{U}_0$. Lemma 12.10 together with the fact that the parameter θ is generic implies that $0 \neq (A(\mu))^{a_1}(\text{Bas}(q_1)) = t(A(\mu))^{a_1}(J(\mu)(v_\infty))$ for some scalar $t \in \mathbb{C}^*$. Since $J(\mu)(v_\infty) \in \mathbf{U}_0$ and the operator $A(\mu)$ lowers degree by one, we have $0 \neq (A(\mu))^{a_1}(\text{Bas}(q_1)) \in \mathbf{U}_{-a_1}$. It now follows from the definition of the matrices $A(\mu)$ and $\Lambda(\mu)$ and the genericity of θ that $(\Lambda(\mu))^{a_1} \circ (A(\mu))^{a_1}(\text{Bas}(q_1)) = t' \text{Bas}(q_1)$ for some scalar $t' \in \mathbb{C}^*$. Since the operator $\Lambda(\mu)$ raises degree by one, we have $\text{Bas}(q_1), \text{Bas}(q_2) + \dots + \text{Bas}(q_k) \in \mathbf{U}_0$.

We can now apply essentially the same argument to $\text{Bas}(q_2)$ and $\text{Bas}(q_2) + \dots + \text{Bas}(q_k)$. By Lemma 12.10 have $0 \neq (A(\mu))^{a_2}(\text{Bas}(q_2)) = t(A(\mu))^{a_2}(\text{Bas}(q_2) + \dots + \text{Bas}(q_k)) + t'(A(\mu))^{a_2}(\text{Bas}(q_1))$ for some scalars $t, t' \in \mathbb{C}^*$. Since $\text{Bas}(q_2) + \dots + \text{Bas}(q_k)$ and $\text{Bas}(q_1)$ are homogeneous elements of degree zero and $A(\mu)$ lowers degree by one, we get $0 \neq (A(\mu))^{a_2}(\text{Bas}(q_2)) \in \mathbf{U}_{-a_2}$. Moreover, Lemma 12.10 implies that $(A(\mu))^{a_2}(\text{Bas}(q_2))$ is a linear combination of $\text{Bas}(q_1 + a_2)$ and $\text{Bas}(q_2 + a_2)$. But $\text{Bas}(q_1 + a_2) = (A(\mu))^{a_2}(\text{Bas}(q_1))$ up to multiplication by a non-zero scalar, so $\text{Bas}(q_1 + a_2) \in \mathbf{U}_{-a_2}$. Hence $\text{Bas}(q_2 + a_2) \in \mathbf{U}_{-a_2}$ and so $\text{Bas}(q_2) = (\Lambda(\mu))^{a_2}(\text{Bas}(q_2 + a_2)) \in \mathbf{U}_0$. We conclude that $\text{Bas}(q_2), \text{Bas}(q_3) + \dots + \text{Bas}(q_k) \in \mathbf{U}_0$. Repeating this argument sufficiently many times shows that $\text{Bas}(q_1), \dots, \text{Bas}(q_k) \in \mathbf{U}_0$. It follows that $\mathbf{U}_0 = \mathbf{W}_0$. Condition (C) and (49) now imply that $\mathbf{U}_i = \mathbf{W}_i$ for all $i \in \mathbb{Z}$. \square

Thanks to Proposition 15.2, we can talk about *the* μ -grading on $\widehat{\mathbf{V}}^\nu$. Let us denote it by $\widehat{\mathbf{V}}^\nu = \bigoplus_{i \in \mathbb{Z}} \mathbf{W}_i^\mu$. We write $\deg_\mu v = i$ if $v \in \mathbf{W}_i^\mu$. Moreover, let $P_\mu := \sum_{i \in \mathbb{Z}} \dim \mathbf{W}_i^\mu t^i$ denote the Poincaré polynomial of \mathbf{V}^ν with respect to the μ -grading. We have a map

$$(\mathcal{X}_\theta(n\delta + \mathbf{d}_\nu))^{\mathbb{C}^*} \rightarrow \mathbb{Z}[t, t^{-1}], \quad [\mathbf{A}(\mu)] \rightarrow P_\mu. \quad (50)$$

Proposition 15.3. *We have $P_\mu = \text{Res}_{\mu^t}(t)$. Moreover, the map (50) is injective.*

Proof. Fix $j \geq 0$. By (48), $\text{Bas}(q_i - j) \in \mathbf{W}_j^\mu$ if and only if $j \leq b_i$, for $i = 1, \dots, k$. Moreover, $\{\text{Bas}(q_i - j) \mid j \leq b_i\}$ form a basis of \mathbf{W}_j^μ . Hence $\dim \mathbf{W}_j^\mu = \sum_{i=1}^k \mathbf{1}_{j \leq b_i}$. Here $\mathbf{1}_{j \leq b_i}$ is the indicator function taking value 1 if $j \leq b_i$ and 0 otherwise. But $\sum_{j=1}^k \mathbf{1}_{i \leq b_j}$ is precisely the number of cells of content $-i$ in μ , which is the same as the number of cells of content i in μ^t . The argument for $j < 0$ is analogous. This proves the first claim. The second claim now follows immediately from the fact that partitions are determined uniquely by their residues. \square

Remark 15.4. Suppose that we are given a \mathbb{C}^* -fixed point and want to find out which partition it is labelled by. Proposition 15.3 implies that to do so we only need to compute the \mathbb{Z} -grading on $\widehat{\mathbf{V}}^\nu$ and the corresponding Poincaré polynomial.

Lemma 15.5. (i) The restricted maps

$$A(\mu) : \mathbf{W}_i^\mu \rightarrow \mathbf{W}_{i-1}^\mu \quad (i \leq 0), \quad \Lambda(\mu) : \mathbf{W}_i^\mu \rightarrow \mathbf{W}_{i+1}^\mu, \quad (i \geq 0)$$

are surjective.

(ii) The restricted maps

$$A(\mu) : \mathbf{W}_i^\mu \rightarrow \mathbf{W}_{i-1}^\mu \quad (i > 0), \quad \Lambda(\mu) : \mathbf{W}_i^\mu \rightarrow \mathbf{W}_{i+1}^\mu \quad (i < 0)$$

are injective.

(iii) We have $\mathbf{V}_j^\nu = \bigoplus_{i \in \mathbb{Z}} \mathbf{W}_{j+il}^\mu$ for each $j = 0, \dots, l-1$.

Proof. The first claim is just a restatement of (49). The second claim follows directly from Lemma 12.10. The third claim follows from (49) and the fact that $A(\mu)$ (resp. $\Lambda(\mu)$) is an operator homogeneous of degree -1 (resp. 1) with respect to the $\mathbb{Z}/l\mathbb{Z}$ -grading on $\widehat{\mathbf{V}}^\nu$. \square

Remark 15.6. We may interpret Lemma 15.5 as saying that the μ -grading on $\widehat{\mathbf{V}}^\nu$ is a *lift* of the $\mathbb{Z}/l\mathbb{Z}$ -grading.

15.4 The reflected grading. Since the reflection functors are \mathbb{C}^* -equivariant, the reflected quiver representation $\mathfrak{R}_i(\mathbf{A}(\mu)) \in \mu_{\sigma_i * \mathbf{d}_\nu}^{-1}(\widetilde{\sigma \cdot \theta})$ is conjugate under the $G(\sigma_i * \mathbf{d}_\nu)$ -action to $\mathbf{A}(\mathbf{R}_i(\mu))$, where $\mathbf{R}_i(\mu) \in \mathcal{P}_{(\sigma_i * \nu)^t}(nl + |(\sigma_i * \nu)^t|)$. The vector space $\widehat{\mathbf{V}}^{\sigma_i * \nu}$ is endowed with the $\mathbf{R}_i(\mu)$ -grading. We now compute this grading.

For $i = 0, \dots, l-1$ set $X_i = \Lambda(\mu)|_{\mathbf{V}_i^\nu}$, $Y_i = A(\mu)|_{\mathbf{V}_i^\nu}$, $I = I(\mu)$ and $J = J(\mu)$. If $i \in \{1, \dots, l-1\}$, we have maps

$$\mathbf{V}_i^\nu \xrightarrow{\psi_i} \mathbf{V}_{i-1}^\nu \oplus \mathbf{V}_{i+1}^\nu \xrightarrow{\phi_i} \mathbf{V}_i^\nu, \quad (51)$$

where $\psi_i = Y_i - X_i$, $\phi_i = X_{i-1} + Y_{i+1}$ (the indices should be considered modulo l). If $i = 0$ we have maps

$$\mathbf{V}_0^\nu \xrightarrow{\psi_0} \mathbf{V}_{l-1}^\nu \oplus \mathbf{V}_1^\nu \oplus \mathbf{V}_\infty^\nu \xrightarrow{\phi_0} \mathbf{V}_0^\nu \quad (52)$$

with $\psi_0 = Y_0 - X_0 + I$ and $\phi_0 = X_{l-1} + Y_1 + J$.

Now let $i \in \{0, \dots, l-1\}$. It follows from the definition of reflection functors (§10.4) that the reflected representation $\mathfrak{R}_i(\mathbf{A}(\mu))$ associates to the vertex i the vector space $\ker \phi_i$ and to any other vertex j the vector space \mathbf{V}_j^ν . This means that $\mathbf{V}_i^{\sigma_i * \nu} \cong \ker \phi_i$.

Let $i \neq 0$. Definition 15.1 implies that there exist direct sum decompositions

$$\psi_i = \bigoplus_{j \in \mathbb{Z}} \psi_i^j, \quad \phi_i = \bigoplus_{j \in \mathbb{Z}} \phi_i^j, \quad (53)$$

with $\psi_i^j = \psi_i|_{\mathbf{W}_{jl+i}^\mu}$ and $\phi_i^j = \phi_i|_{\mathbf{W}_{jl+i-1}^\mu \oplus \mathbf{W}_{jl+i+1}^\mu}$. Hence (51) decomposes as a direct sum of maps

$$\mathbf{W}_{jl+i}^\mu \xrightarrow{\psi_i^j} \mathbf{W}_{jl+i-1}^\mu \oplus \mathbf{W}_{jl+i+1}^\mu \xrightarrow{\phi_i^j} \mathbf{W}_{jl+i}^\mu \quad (j \in \mathbb{Z}).$$

If $i = 0$ then, by Lemma 15.1, we also have decompositions (53) with $\psi_0^j = \psi_0|_{\mathbf{W}_{jl}^\mu}$ and $\phi_0^j = \phi_0|_{\mathbf{W}_{jl-1}^\mu \oplus \mathbf{W}_{jl+1}^\mu}$ for $j \neq 0$ and $\psi_0^0 = \psi_0|_{\mathbf{W}_0^\mu}$, $\phi_0^0 = \phi_0|_{\mathbf{W}_{-1}^\mu \oplus \mathbf{W}_1^\mu \oplus \mathbf{V}_\infty}$ for $j = 0$. Hence (52) decomposes as a direct sum of maps

$$\begin{aligned} \mathbf{W}_{jl}^\mu &\xrightarrow{\psi_0^j} \mathbf{W}_{jl-1}^\mu \oplus \mathbf{W}_{jl+1}^\mu \xrightarrow{\phi_0^j} \mathbf{W}_{jl}^\mu \quad (j \in \mathbb{Z} - \{0\}), \\ \mathbf{W}_0^\mu &\xrightarrow{\psi_0^0} \mathbf{W}_{-1}^\mu \oplus \mathbf{W}_1^\mu \oplus \mathbf{V}_\infty \xrightarrow{\phi_0^0} \mathbf{W}_0^\mu. \end{aligned}$$

These direct sum decompositions together with the preprojective relations imply the following lemma.

Lemma 15.7. *Let $i \in \{0, \dots, l-1\}$. Then $\ker \phi_i = \bigoplus_{j \in \mathbb{Z}} \ker \phi_i^j$. If $i = 0$ and $j = 0$ then $\text{Im } \psi_0^0 \oplus \ker \phi_0^0 = \mathbf{W}_{-1}^\mu \oplus \mathbf{W}_1^\mu \oplus \mathbf{V}_\infty$. Otherwise $\text{Im } \psi_i^j \oplus \ker \phi_i^j = \mathbf{W}_{jl+i-1}^\mu \oplus \mathbf{W}_{jl+i+1}^\mu$.*

Set

$$\mathbf{U}_j := \mathbf{W}_j^\mu \quad (j \neq i \bmod l), \quad \mathbf{U}_{jl+i} := \ker \phi_i^j \quad (j \in \mathbb{Z}).$$

Proposition 15.8. *The \mathbb{Z} -grading $\widehat{\mathbf{V}}^{\sigma_i * \nu} = \bigoplus_{j \in \mathbb{Z}} \mathbf{U}_j$ is the $\mathbf{R}_i(\mu)$ -grading on $\widehat{\mathbf{V}}^{\sigma_i * \nu}$.*

Proof. It suffices to show that $\widehat{\mathbf{V}}^{\sigma_i * \nu} = \bigoplus_{j \in \mathbb{Z}} \mathbf{U}_j$ satisfies condition (C) in Definition 15.1. Suppose $i \neq 0$. Then for each $j \in \mathbb{Z}$ we need to check that

$$A(\mathbf{R}_i(\mu))(\mathbf{U}_{jl+i+1}) \subseteq \mathbf{U}_{jl+i}, \quad \Lambda(\mathbf{R}_i(\mu))(\mathbf{U}_{jl+i-1}) \subseteq \mathbf{U}_{jl+i},$$

$$A(\mathbf{R}_i(\mu))(\mathbf{U}_{jl+i}) \subseteq \mathbf{U}_{jl+i-1}, \quad \Lambda(\mathbf{R}_i(\mu))(\mathbf{U}_{jl+i}) \subseteq \mathbf{U}_{jl+i+1}.$$

If $i = 0$ we additionally need to check that $I(\mathbf{R}_i(\mu))(\mathbf{U}_0) = \mathbf{V}_\infty$ and $J(\mathbf{R}_i(\mu))(\mathbf{V}_\infty) \subseteq \mathbf{U}_0$.

All of the inclusions above follow directly from Lemma 15.7 and the definition of reflection functors in §10.4. For example, let us assume that $i \neq 0$ or $j \neq 0$ and consider the inclusion $A(\mathbf{R}_i(\mu))(\mathbf{U}_{jl+i+1}) \subseteq \mathbf{U}_{jl+i} := \ker \phi_i^j$. By the definition of reflection functors, the map $A(\mathbf{R}_i(\mu)) : \mathbf{V}_{i+1}^{\sigma_i * \nu} = \mathbf{V}_{i+1}^\nu \rightarrow \mathbf{V}_i^{\sigma_i * \nu} = \ker \phi$ is defined as the composition

$$\mathbf{V}_{i+1}^\nu \xrightarrow{\cdot(-\theta_i)} \mathbf{V}_{i+1}^\nu \hookrightarrow \mathbf{V}_{i-1}^\nu \oplus \mathbf{V}_{i+1}^\nu = \ker \phi \oplus \text{Im } \psi \twoheadrightarrow \ker \phi \xrightarrow{\cdot(-1)} \ker \phi.$$

Consider the restriction of this map to the subspace $\mathbf{U}_{jl+i+1} = \mathbf{W}_{jl+i+1}^\mu \subseteq \mathbf{V}_{i+1}^\nu$. By Lemma 15.7 we have $\mathbf{W}_{jl+i+1}^\mu \subseteq \mathbf{W}_{jl+i-1}^\mu \oplus \mathbf{W}_{jl+i+1}^\mu = \ker \phi_i^j \oplus \text{Im } \psi_i^j$. Hence $A(\mathbf{R}_i(\mu))|_{\mathbf{W}_{jl+i+1}^\mu}$ is the composition

$$\mathbf{W}_{jl+i+1}^\nu \xrightarrow{\cdot(-\theta_i)} \mathbf{W}_{jl+i+1}^\nu \hookrightarrow \mathbf{W}_{jl+i-1}^\nu \oplus \mathbf{W}_{jl+i+1}^\nu = \ker \phi_i^j \oplus \text{Im } \psi_i^j \twoheadrightarrow \ker \phi_i^j \xrightarrow{\cdot(-1)} \ker \phi_i^j.$$

In particular, $A(\mathbf{R}_i(\mu))(\mathbf{W}_{jl+i+1}^\nu) = \ker \phi_i^j$ as desired. The other inclusions are proven in an analogous way. \square

15.5 The dimension formula. We have described the $\mathbf{R}_i(\mu)$ -grading on $\widehat{\mathbf{V}}^{\sigma_i * \nu}$. We now want to compute its Poincaré polynomial $P_{\mathbf{R}_i(\mu)}$.

Lemma 15.9. *Let $\widehat{\mathbf{V}}^\nu = \bigoplus_{i \in \mathbb{Z}} \mathbf{W}_i^\mu$ be the μ -grading on $\widehat{\mathbf{V}}^\nu$. Then*

$$\dim \ker \phi_i^j = \dim \mathbf{W}_{lj+i+1}^\mu + \dim \mathbf{W}_{lj+i-1}^\mu - \dim \mathbf{W}_{lj+i}^\mu$$

if $j \neq 0$ or $i \neq 0$. Otherwise

$$\dim \ker \phi_0^0 = \dim \mathbf{W}_1^\mu + \dim \mathbf{W}_{-1}^\mu - \dim \mathbf{W}_0^\mu + 1.$$

Proof. Assume that $j \neq 0$ or $i \neq 0$. Recall that $\psi_i^j = A(\mu)|_{\mathbf{W}_{jl+i}^\nu} - \Lambda(\mu)|_{\mathbf{W}_{jl+i}^\nu}$. Lemma 15.5 implies that either $A(\mu)|_{\mathbf{W}_{jl+i}^\nu}$ or $\Lambda(\mu)|_{\mathbf{W}_{jl+i}^\nu}$ is injective. Hence ψ_i^j is injective. Therefore we have $\dim \text{Im } \psi_i^j = \dim \mathbf{W}_{lj+i}^\mu$. The equality $\dim \ker \phi_i^j = \dim \mathbf{W}_{lj+i+1}^\mu + \dim \mathbf{W}_{lj+i-1}^\mu - \dim \text{Im } \psi_i^j$ now implies the lemma. The case $i = j = 0$ is similar. \square

Corollary 15.10. *Write $P_\mu = \sum_{j \in \mathbb{Z}} a_j^\mu t^j$ with $a_j^\mu = \dim \mathbf{W}_j^\mu$ and $P_{\mathbf{R}_i(\mu)} = \sum_{j \in \mathbb{Z}} a_j^{\mathbf{R}_i(\mu)} t^j$ with $a_j^{\mathbf{R}_i(\mu)} = \dim \mathbf{W}_j^{\mathbf{R}_i(\mu)}$. Then $a_j^\mu = a_j^{\mathbf{R}_i(\mu)}$ for $j \neq i \bmod l$. Moreover $a_{lj+i}^{\mathbf{R}_i(\mu)} = a_{lj+i+1}^\mu + a_{lj+i-1}^\mu - a_{lj+i}^\mu$ if $j \neq 0$ or $i \neq 0$ and $a_0^{\mathbf{R}_0(\mu)} = a_1^\mu + a_{-1}^\mu - a_0^\mu + 1$ otherwise.*

Proof. This follows directly from Proposition 15.8 and Lemma 15.9. \square

15.6 Removable and addable cells. Throughout this subsection let μ be an arbitrary partition. Recall the partition $\mathbf{T}_i(\mu)$ (see §10.2) obtained from μ through adding all i -addable cells and removing all i -removable cells. We will now interpret addability and removability in terms of the residue of μ .

Definition 15.11. Let $k \in \mathbb{Z}$. We say that the Young diagram $\mathbb{Y}(\mu)$ has a *corner* at k if $\mathbb{Y}(\mu)$ contains a cell of content k and the unique cell (i, j) of content k which lies on the rim of $\mathbb{Y}(\mu)$ has the property that $(i+1, j), (i, j+1) \notin \mathbb{Y}(\mu)$. We call that cell the k -corner. We say that $\mathbb{Y}(\mu)$ has a *niche* (or cocorner) at k if the unique cell (i, j) of content k which lies on the rim of $\mathbb{Y}(\mu)$ has the property that $(i+1, j), (i, j+1) \in \mathbb{Y}(\mu)$. We call that cell the k -niche. Additionally, if $\lambda = (\lambda_1, \dots, \lambda_m)$ we say that $\mathbb{Y}(\mu)$ has a niche at λ_1 and $-m$. Finally, we say that $\mathbb{Y}(\mu)$ has a *wall* at k if it has neither a corner nor a niche at k .

Remark 15.12. Recall that a finite subset Y of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ is a Young diagram if and only if it satisfies the property that for any $(i, j) \in Y$, $i > 1$ implies that $(i-1, j) \in Y$ and $j > 1$ implies $(i, j-1) \in Y$.

We will need the following somewhat technical lemma.

Lemma 15.13. Let us write $\text{Res}_\mu(t) = \sum_{j \in \mathbb{Z}} b_j t^j$.

(i) The following are equivalent: (1) $\mathbb{Y}(\mu)$ has a corner at k ; (2) a cell of content k is removable relative to μ ; (3) $b_{k-1} = b_k, b_{k+1} = b_k - 1$ ($k > 0$) or $b_{k+1} = b_k, b_{k-1} = b_k - 1$ ($k < 0$) or $b_{-1} = b_1, b_0 = b_1 + 1$ ($k = 0$).

(ii) The following are equivalent: (1) $\mathbb{Y}(\mu)$ has a niche at k ; (2) a cell of content k is addable relative to μ ; (3) $b_{k+1} = b_k, b_{k-1} = b_k + 1$ ($k > 0$) or $b_{k-1} = b_k, b_{k+1} = b_k + 1$ ($k < 0$) or $b_{-1} = b_0 = b_1$ ($k = 0$).

(iii) The following are equivalent: (1) $\mathbb{Y}(\mu)$ has a wall at i ; (2) no cell of content k is addable or removable relative to μ ; (3) $[b_{k+1} = b_k = b_{k-1} \text{ or } b_{k+1} = b_k - 1 = b_{k-1} - 2 \text{ } (k > 0)]$, or $[b_{k+1} = b_k = b_{k-1} \text{ or } b_{k+1} = b_k + 1 = b_{k-1} + 2 \text{ } (k < 0)]$, or $[b_{-1} = b_0 = b_1 + 1 \text{ or } b_1 = b_0 = b_{-1} + 1 \text{ } (k = 0)]$.

Proof. (i) Suppose that $\mathbb{Y}(\mu)$ does not have a corner at k . Then either $\mathbb{Y}(\mu)$ doesn't contain a cell of content k or the unique cell (i, j) of content k which lies on the rim of $\mathbb{Y}(\mu)$ has the property that $(i+1, j) \in \mathbb{Y}(\mu)$ or $(i, j+1) \in \mathbb{Y}(\mu)$. If $\mathbb{Y}(\mu)$ doesn't contain a cell of content k then trivially no cell of content k can be removed from $\mathbb{Y}(\mu)$. In the second case removing (i, j) clearly results in a shape which is not a Young diagram of a partition. Conversely, if $\mathbb{Y}(\mu)$ has a corner at k then removing the k -corner preserves the condition in Remark 15.12, so the k -corner is removable. This proves the equivalence of (1) and (2).

Let $k > 0$. Suppose that the cell (i, j) is removable from $\mathbb{Y}(\mu)$ and has content k . Then $j - i = k$ and $(1, k+1), (2, k+2), \dots, (i, j)$ are precisely the cells of content k in $\mathbb{Y}(\mu)$. Since $\mathbb{Y}(\mu)$ has a corner at k , the cells of content $k-1$ in $\mathbb{Y}(\mu)$ are precisely $(1, k), (2, k+1), \dots, (i, j-1)$ and the cells of content $k+1$ in $\mathbb{Y}(\mu)$ are precisely $(1, k+2), (2, k+3), \dots, (i-1, j)$. Hence $b_k = i, b_{k-1} = i, b_{k+1} = i-1$, which yields the desired equalities. Conversely, suppose that $b_{k-1} = b_k, b_{k+1} = b_k - 1$. Then the cell $(b_k, b_k + k)$ is removable relative to μ . Indeed, $b_{k-1} = b_k$ implies that $(b_k + 1, b_k + k) \notin \mathbb{Y}(\mu)$ and $b_{k+1} = b_k - 1$ implies that $(b_k, b_k + k + 1) \notin \mathbb{Y}(\mu)$.

The proofs of the remaining cases are similar so we omit them. \square

15.7 Combinatorial interpretation of reflection functors. We can now interpret the effect of applying reflection functors to the fixed points combinatorially.

Proposition 15.14. Let $\mu \in \mathcal{P}_{\nu^t}(nl + |\nu^t|)$. We have $\mathbf{R}_k(\mu) = (\mathbf{T}_k(\mu^t))^t$.

Proof. It suffices to show that the residue of $\mathbf{T}_k(\mu^t)$ equals $P_{\mathbf{R}_k(\mu)}$. Let us write $\text{Res}_{(\mathbf{T}_k(\mu^t))}(t) = \sum_{i \in \mathbb{Z}} a_i t^i$, $\text{Res}_{\mu^t}(t) = \sum_{i \in \mathbb{Z}} b_i t^i$ and $P_{\mathbf{R}_k(\mu)} = \sum_{i \in \mathbb{Z}} c_i t^i$, $P_\mu = \sum_{i \in \mathbb{Z}} d_i t^i$. By Lemma 15.1 we have $b_i = d_i$.

Suppose that $i \neq k \bmod l$. Then $a_i = b_i = d_i = c_i$. The first equality follows from the fact that no cells of content i are added to or removed from μ^t when we transform μ^t into $\mathbf{T}_k(\mu^t)$. The third equality follows from Corollary 15.10.

Now suppose that $i = k \bmod l$ and $i > 0$. Then $c_i = d_{i+1} + d_{i-1} - d_i$ by Corollary 15.10. Hence $c_i = b_{i+1} + b_{i-1} - b_i$. We now argue that $b_{i+1} + b_{i-1} - b_i = a_i$. There are three possibilities: $a_i = b_i + 1$ and one cell of content i is addable to μ^t , or $a_i = b_i - 1$ and one cell of content i is removable from μ^t , or $a_i = b_i$ and no cell of content i is addable to or removable from μ^t . In the first case we have $b_i = b_{i+1}$ and $b_{i-1} = b_i + 1$. In the second case we have $b_i = b_{i-1}$ and $b_{i+1} = b_i - 1$. In the third case we have $b_{i-1} = b_i = b_{i+1}$ or $b_i = b_{i+1} + 1 = b_{i-1} - 1$. These equalities follow immediately from Lemma 15.13. In each of the three cases we see that the equality $b_{i+1} + b_{i-1} - b_i = a_i$ holds. Hence $a_i = c_i$. The proof for $i \leq 0$ is analogous. \square

Recall that if $\lambda \in \mathcal{P}$ then

$$\underline{\text{Quot}}(\lambda^t) = ((\underline{\text{Quot}}(\lambda))^t)^\flat. \quad (54)$$

Corollary 15.15. *Let $\mu \in \mathcal{P}_{\nu^t}(nl + |\nu^t|)$ and $i \in \{0, \dots, l-1\}$. Then $\mathbf{R}_i(\mu) = (\sigma_i * \mu^t)^t$. Moreover,*

$$\text{Core}(\mathbf{R}_i(\mu)) = (\sigma_i * \nu)^t = (\mathbf{T}_i(\nu))^t, \quad \underline{\text{Quot}}(\mathbf{R}_i(\mu)) = s_{l-i} \cdot \underline{\text{Quot}}(\mu).$$

Proof. The first claim follows directly from Proposition 15.14 and the definition of the \tilde{S}_l -action on partitions in §10.2. The formula for $\text{Core}(\mathbf{R}_i(\mu))$ follows directly from Proposition 10.3. The formula for $\underline{\text{Quot}}(\mathbf{R}_i(\mu))$ follows from Proposition 10.3 and (54). Indeed,

$$\begin{aligned} \underline{\text{Quot}}(\mathbf{R}_i(\mu)) &= \underline{\text{Quot}}((\sigma_i * \mu^t)^t) = ((\underline{\text{Quot}}(\sigma_i * \mu^t))^t)^\flat \\ &= ((s_i \cdot \underline{\text{Quot}}(\mu^t))^t)^\flat \\ &= (s_i \cdot (\underline{\text{Quot}}(\mu))^\flat)^\flat = s_{l-i} \cdot \underline{\text{Quot}}(\mu). \end{aligned}$$

\square

Note that by Proposition 10.3 we also have

$$\text{Core}((\mathbf{R}_i(\mu))^t) = \sigma_i * \nu = \mathbf{T}_i(\nu), \quad \underline{\text{Quot}}((\mathbf{R}_i(\mu))^t) = \text{pr}(\sigma_i) \cdot \underline{\text{Quot}}(\mu) = s_i \cdot \underline{\text{Quot}}(\mu). \quad (55)$$

15.8 Rotation of complex structure and embedding into $\text{Hilb}(K)$. We have described how the fixed points behave under reflection functors. We now need to investigate their behaviour under the map

$$\Phi : \mathcal{X}_{-\frac{1}{2}}(\gamma) \rightarrow \mathcal{M}_{-1}(\gamma) \rightarrow \text{Hilb}(\nu) \quad (56)$$

from §10.6. It induces a bijection between the \mathbb{C}^* -fixed points and hence also a bijection between their labelling sets

$$\Psi : \mathcal{P}_{\nu^t}(nl + |\nu^t|) \rightarrow \mathcal{P}_\nu(nl + |\nu|), \quad \mu \mapsto \lambda,$$

where the partition λ is defined by the equation $I_\lambda = \Phi([\mathbf{A}(\mu)])$.

Proposition 15.16. *We have $\Psi(\mu) = \mu^t$.*

Proof. Let $\mathcal{V}_{-\frac{1}{2}}(\gamma) := \mu_{\mathbf{d}_\nu}^{-1}(\widetilde{-\frac{1}{2}}) \times^{G(\mathbf{d}_\nu)} \widehat{\mathbf{V}}^\nu$ denote the tautological bundle on $\mathcal{X}_{-\frac{1}{2}}(\gamma)$. The diffeomorphism (56) lifts to a $U(1)$ -equivariant isomorphism of tautological vector bundles

$$\mathcal{V}_{-\frac{1}{2}}(\gamma) \xrightarrow{\cong} \mathcal{T}(\nu), \quad (57)$$

where $\mathcal{T}(\nu)$ denotes the restriction of $\mathcal{T}(K)$ to the subscheme $\text{Hilb}(\nu)$. Let $\mu \in \mathcal{P}_{\nu^t}(nl + |\nu^t|)$. Proposition 12.15 implies that $\text{ch}_t \left(\mathcal{V}_{-\frac{1}{2}}(\gamma)_{[\mathbf{A}(\mu)]} \right) = \text{Res}_\mu(t)$. It follows that the \mathbb{C}^* -characters of the fibres of $\mathcal{V}_{-\frac{1}{2}}(\gamma)$ at any two distinct \mathbb{C}^* -fixed points are distinct. By Lemma 10.5 we have $\text{ch}_t(\mathcal{T}(\nu)_{I_\mu}) = \text{Res}_{\mu^t}(t)$. The $U(1)$ -equivariance of (57) implies that $\Phi([\mathbf{A}(\mu)]) = I_{\mu^t}$ and so $\Psi(\mu) = \mu^t$. \square

15.9 Matching the \mathbb{C}^* -fixed points. We are now ready to collect all our results about the \mathbb{C}^* -fixed points. Let $w \in \tilde{S}_l$ and set $\theta := w^{-1} \cdot (-\frac{1}{2}) \in \mathbb{Q}^l$ as well as $\gamma := w * n\delta \in \mathbb{Z}^l$. We have $\gamma = n\delta + \gamma_0$, where $\gamma_0 = w * \emptyset$. Let $\nu := \mathfrak{d}^{-1}(\tau_{\gamma_0})$ be the l -core corresponding to γ_0 . We choose a reduced expression $w = \sigma_{i_1} \dots \sigma_{i_m}$ for w in \tilde{S}_l . Set $\mathbf{h} := (h, H_1, \dots, H_{l-1})$, where $H_j = \theta_j$ ($1 \leq j \leq l-1$) and $h = -\theta_0 - \sum_{j=1}^{l-1} H_j$. Composing the Etingof-Ginzburg map with (24) and (27) we obtain a $U(1)$ -equivariant (non-algebraic) isomorphism

$$\mathrm{Spec} Z_{0,\mathbf{h}} \xrightarrow{\mathrm{EG}} \mathcal{X}_\theta(n\delta) \xrightarrow{\mathfrak{R}_{i_1} \circ \dots \circ \mathfrak{R}_{i_m}} \mathcal{X}_{-\frac{1}{2}}(\gamma) \xrightarrow{\Phi} \mathrm{Hilb}(\nu). \quad (58)$$

The isomorphism (58) induces bijections between the labelling sets of the \mathbb{C}^* -fixed points. The following theorem is our second main result.

Theorem 15.17. *The map (58) induces the following bijections*

$$\begin{array}{ccccccc} \mathcal{P}(l, n) & \longrightarrow & \mathcal{P}_\emptyset(nl) & \longrightarrow & \mathcal{P}_{\nu^t}(nl + |\nu^t|) & \longrightarrow & \mathcal{P}_\nu(nl + |\nu|) \\ (\underline{\mathrm{Quot}}(\mu))^b & \longmapsto & \mu & \longmapsto & (w * \mu^t)^t & \longmapsto & w * \mu^t. \end{array}$$

Moreover,

$$\nu = w * \emptyset = \mathbf{T}_{i_1} \circ \dots \circ \mathbf{T}_{i_m}(\emptyset), \quad \underline{\mathrm{Quot}}(w * \mu^t) = \mathrm{pr}(w) \cdot \underline{\mathrm{Quot}}(\mu^t).$$

Proof. The theorem just brings together the results of Corollary 14.17, Proposition 15.16 and (55). \square

Given $w \in \tilde{S}_l$ and \mathbf{h} as above (note that \mathbf{h} depends on w), we define the \mathbf{h} -twisted l -quotient bijection to be the map

$$\tau_{\mathbf{h}} : \mathcal{P}(l, n) \rightarrow \mathcal{P}_\nu(nl + |\nu|), \quad \underline{\mathrm{Quot}}(\mu) \mapsto w * \mu.$$

Using this terminology, we can reformulate Theorem 15.17 in the following way. Let $\Omega : \mathcal{P}(l, n) \rightarrow \mathcal{P}_\nu(nl + |\nu|)$ denote the bijection between the labelling sets of the \mathbb{C}^* -fixed points induced by (58).

Corollary 15.18. *We have*

$$\Omega(\underline{\lambda}) = \tau_{\mathbf{h}}(\underline{\lambda}^t).$$

Proof. Suppose that $\underline{\lambda} = (\underline{\mathrm{Quot}}(\mu))^b$. Then Theorem 15.17 implies that $\Omega(\underline{\lambda}) = w * \mu^t$. On the other hand, $\underline{\lambda}^t = ((\underline{\mathrm{Quot}}(\mu))^b)^t = \underline{\mathrm{Quot}}(\mu^t)$ by 54. Hence $\tau_{\mathbf{h}}(\underline{\lambda}^t) = w * \mu^t$. \square

With Corollary 15.18 we have achieved our initial goal - the proof of Claim B from the introduction.

15.10 Extension to all regular parameters. Above we made the assumption that the parameter θ lies in the \tilde{S}_l -orbit of the parameter $-\frac{1}{2}$. We now recall how to extend our results to all regular parameters. Recall the \tilde{S}_l -action on the parameter space \mathbb{Q}^l from §10.1. Observe that $-h = \sum_{i=0}^{l-1} \theta_i$ is preserved under this action. Hence \tilde{S}_l acts on $\Theta_{-\frac{1}{2}} := \{(\theta_0, \dots, \theta_{l-1}) \in \mathbb{Q}^l \mid \sum_{i=0}^{l-1} \theta_i = -1/2\}$. Using [10, Lemma 7.1] and the fact that the quiver varieties $\mathcal{X}_\theta(n\delta)$ and $\mathcal{M}_{2\theta}(n\delta)$ are invariant under scaling θ by $\mathbb{Q}_{>0}$, it suffices to consider the parameters in $\Theta_{-\frac{1}{2}}$. Let $\mathbb{Q}_{reg}^l := \{\theta \in \mathbb{Q}^l \mid \mathcal{M}_{2\theta}(n\delta) \text{ is smooth}\}$. The hyperplanes defined in Proposition 3.4 partition \mathbb{Q}_{reg}^l into GIT chambers. Set $\Theta_{-\frac{1}{2}}^{reg} := \mathbb{Q}_{reg}^l \cap \Theta_{-\frac{1}{2}}$. The decomposition of \mathbb{Q}_{reg}^l into GIT chambers induces a corresponding decomposition of $\Theta_{-\frac{1}{2}}^{reg}$. The reflecting hyperplanes of the \tilde{S}_l -action on $\Theta_{-\frac{1}{2}}$ also partition $\Theta_{-\frac{1}{2}}^{reg}$ into alcoves. By [10, Lemma 7.2], the decomposition of $\Theta_{-\frac{1}{2}}^{reg}$ into alcoves refines the decomposition into GIT chambers. But the \tilde{S}_l -action on the alcoves is transitive. It follows that we can reach every GIT chamber from the parameter $-\frac{1}{2}$ using the \tilde{S}_l -action.

15.11 Geometric ordering. We are now going to relate the matching of the \mathbb{C}^* -fixed points with the geometric ordering, thereby finishing the proof of Claim A from the introduction. Let θ and \mathbf{h} be as in §15.9. By [10, §3.9] there exists a symplectic resolution $\pi_\theta : \mathcal{M}_{2\theta}(n\delta) \rightarrow (\mathfrak{h} \oplus \mathfrak{h}^*)/\Gamma_n$. Moreover, by [28, Proposition 3] the Hilbert-Chow morphism $\text{Hilb}(K) \rightarrow S^K(\mathbb{C}^2)$ restricts to a morphism $\pi : \text{Hilb}(\nu) \rightarrow (\mathfrak{h} \oplus \mathfrak{h}^*)/\Gamma_n$. We thus have a commutative diagram

$$\begin{array}{ccccccc} \text{Spec } Z_{0,\mathbf{h}} & \xrightarrow{\sim} & \mathcal{X}_\theta(n\delta) & \longrightarrow & \mathcal{M}_{2\theta}(n\delta) & \xrightarrow{\sim} & \text{Hilb}(\nu) \\ & & & & \searrow \pi_\theta & & \swarrow \pi \\ & & & & & & (\mathfrak{h} \oplus \mathfrak{h}^*)/\Gamma_n \end{array}$$

The first two horizontal maps are, as before, the Etingof-Ginzburg map and the rotation of complex structure. The third horizontal map arises as a composition of reflection functors. For $\underline{\lambda} \in \mathcal{P}(l, n)$ let $m_{\underline{\lambda}}$ be the image of $\text{Ann}(\underline{\lambda}) \in \text{Spec } Z_{0,\mathbf{h}}$ under the Etingof-Ginzburg map and the rotation of complex structure. Set $\mathcal{Z}_\theta := \pi_\theta^{-1}(\{0\} \times \mathfrak{h}^*/\Gamma_n)$ and let

$$\mathcal{Z}_{\underline{\lambda}} := \{x \in \mathcal{M}_\theta(n\delta) \mid \lim_{t \rightarrow 0} t.x = m_{\underline{\lambda}}\}.$$

be the attracting set of the fixed point $m_{\underline{\lambda}}$. By [10, Lemma 5.4], $\mathcal{Z}_\theta = \bigsqcup_{\underline{\lambda} \in \mathcal{P}(l, n)} \mathcal{Z}_{\underline{\lambda}}$. The geometric ordering on $\mathcal{P}(l, n)$ is defined to be the partial order generated by the rule

$$\underline{\mu} \preceq_{\mathbf{h}} \underline{\lambda} \iff \overline{\mathcal{Z}_{\underline{\lambda}}} \cap \mathcal{Z}_{\underline{\mu}} \neq \emptyset.$$

Using the Hilbert-Chow map π , one can define an analogous stratification on $\text{Hilb}(\nu)$ and a corresponding geometric ordering on $\mathcal{P}_\nu(nl + |\nu|)$. By construction, the isomorphism $\mathcal{M}_{2\theta}(n\delta) \xrightarrow{\sim} \text{Hilb}(\nu)$ intertwines the two geometric orderings. Moreover, by [21] the geometric ordering coincides with the anti-dominance ordering on $\mathcal{P}_\nu(nl + |\nu|)$. It follows that

$$\underline{\mu} \preceq_{\mathbf{h}} \underline{\lambda} \iff \Omega(\underline{\lambda}) \leq \Omega(\underline{\mu}) \iff \tau_{\mathbf{h}}(\underline{\lambda}^t) \leq \tau_{\mathbf{h}}(\underline{\mu}^t),$$

where \leq denotes the dominance ordering. This result can be extended to an arbitrary regular parameter θ as explained in §15.10. We have thus established Claim A from the introduction.

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